Note: Please insert your assignment into the mailboxes 5th floor, building 46. The problems marked by a ☗ have to be submitted.

☗ **Problem 23.** *Coupled oscillators; normal coordinates*

The eigenvalues of a single harmonic oscillator were derived in the lecture. Now consider two coupled oscillators

$$\hat{H} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{1}{2} m\omega^2 \left(\hat{x}_1^2 + \hat{x}_2^2\right) + \gamma m\omega^2 \hat{x}_1 \hat{x}_2, \tag{1}$$

where $0 < \gamma < 1$ characterizes the coupling strength.

(a) Determine the eigenvalues for a vanishing coupling strength ($\gamma = 0$). What is the degree of degeneracy of the corresponding eigenfunctions, i.e. how many eigenfunctions for the same eigenvalue exist?

(b) Introduce new variables ξ and η and write $x_1 = \frac{1}{\sqrt{2}}(\xi + \eta)$ and $x_2 = \frac{1}{\sqrt{2}}(\xi - \eta)$. What is the Hamiltonian in the new coordinates and what are the corresponding momenta?

(c) Determine the eigenvalues for $\gamma \neq 0$. Show that the coupling term in general removes the degeneracy found in (a).

☗ **Problem 24.** *Coherent states*

(a) Show that coherent states are normalized, i.e. $\langle\alpha|\alpha\rangle = 1$, but are not orthogonal. Determine $\langle\alpha|\beta\rangle$.

(b) Use the Baker-Hausdorff theorem to show that coherent states satisfy:

$$|\alpha\rangle = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)|0\rangle\,, \tag{2}$$

where $|0\rangle$ is the eigenstate of $\hat{n} = \hat{a}^\dagger\hat{a}$ for the eigenvalue 0.

(c) Show that the coherent states are overcomplete

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\; |\alpha\rangle\langle\alpha| = \mathbb{1}, \tag{3}$$

where $\int \mathrm{d}^2\alpha = \int \mathrm{d}u \int \mathrm{d}v = \int_0^\infty \mathrm{d}r\; r \int_0^{2\pi} \mathrm{d}\varphi$ with $\alpha = u + iv = re^{i\varphi}$.
Hint: Use $\int_0^\infty \mathrm{d}x\; x^n e^{-x} = n!$.

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Problem 25. *Angular momentum; ladder operators*

The matrices

$$\hat{L}_x \equiv \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \hat{L}_y \equiv \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \hat{L}_z = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

are the spatial components of the angular momentum for $l = 1$ in the basis of eigenstates $|m = -1\rangle\,, |m = 0\rangle\,, |m = 1\rangle$ of $\hat{L}_z$.

(a) Show that the eigenvalue of $\hat{\vec{L}}^2$ is $\hbar^2 l(l+1)$ with $l = 1$.

(b) Determine the eigenvectors of $\hat{L}_x, \hat{L}_y$ and $\hat{L}_z$ for the eigenvalue $m = 0$. Show that this set of vectors yields an orthogonal complete basis.