Note: Please insert your assignment into the mailboxes 5th floor, building 46. The problems marked by a ☗ have to be submitted.

☗ **Problem 20.** *Half harmonic oscillator*
Consider a harmonic oscillator in one dimension with frequency ω and mass m. The corresponding energy eigenvalues and functions are E_n and $\phi_n(x)$. An infinite potential well is added at $x = 0$

$$V(x) = \begin{cases} \infty & \text{for } x \leq 0 \\ \frac{m\omega^2}{2}x^2 & \text{for } x > 0. \end{cases}$$

(a) Explain why the eigenfunction of the harmonic oscillator with the potential barrier, besides a normalization factor, is a subset of eigenfunctions ϕ_n of the harmonic oszillator. What are the corresponding eigenenergies?

(b) Let $\Psi_0(x)$ be an arbitrary wave function in the potential V(x) at time $t = 0$. Show that at time $t = T$ the wave function evolves into its negative, i.e.

$$\Psi(x, T) = -\Psi(x, 0) = -\Psi_0(x).$$

T is the period of the oscillation $\omega T = 2\pi$. What happens at $t = 2T$?

☗ **Problem 21.** *Sum rule*
Consider a particle in a one dimensional Potential $V(x)$ which has bound states $|E_n\rangle$ to the eigenvalues E_n of the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$
$$\hat{H}|E_n\rangle = E_n |E_n\rangle .$$

(a) Show that

$$\left[\left[\hat{x}, \hat{H}\right], \hat{x}\right] = \frac{\hbar^2}{m}.$$

(b) Use (a) and the identity operator $\mathbb{1} = \sum_k |E_k\rangle\langle E_k|$, prove the sum rule of the dipole matrix element $\hat{d} = e\hat{x}$

$$\sum_k \omega_{kn} |\langle E_n| \hat{d} |E_k\rangle|^2 = \sum_k \omega_{kn} \langle E_n| \hat{d} |E_k\rangle \langle E_k| \hat{d} |E_n\rangle = \frac{\hbar e^2}{2m},$$

where $\hbar\omega_{kn} = E_k - E_n$.

(c) Check explicitly the sum rule for the one-dimensional harmonic oscillator.

Problem 22. *Hermite Polynomials*
Prove that $e^{2\lambda\xi-\lambda^2}$ is a generating function of the Hermite polynomials , i.e.

$$e^{2\lambda\xi-\lambda^2} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(\xi).$$

Show the symmetry property $H_n(-\xi) = (-1)^n H_n(\xi)$ and that the Hermite polynomials satisfy

$$H_n^{'}(\xi) = 2nH_{n-1}(\xi),$$
$$2\xi H_n(\xi) = 2nH_{n-1}(\xi) + H_{n+1}(\xi).$$