Note: Please insert your assignment into the mailboxes 5th floor, building 46.

Problem 14. *Wave packet*
Consider a wave packet in one dimension

$$\widetilde{\psi}(k,0) = A\mathrm{e}^{-\frac{k^2}{4\sigma_k^2}}.$$

Verify the following results from the lecture:

$$\psi(x,0) = \frac{A\sigma_k}{\sqrt{\pi}}\mathrm{e}^{-x^2\sigma_k^2} = \frac{A}{2\sigma_x\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{4\sigma_x^2}},$$

where $\sigma_x = 1/2\sigma_k$ and

$$\psi(x,t) = \frac{A}{2\sqrt{\pi}\sqrt{\sigma_x^2 + \frac{i\hbar t}{2m}}}\mathrm{e}^{-\frac{x^2}{4(\sigma_x^2+\frac{i\hbar t}{2m})}}.$$

Show further

$$\begin{aligned}\Delta x(t)^2 =& \Delta x(0)^2 + \frac{\hbar^2 t^2}{4m^2\Delta x(0)^2},\\ \Delta p(t)^2 =& \Delta p(0)^2.\end{aligned}$$

Problem 15. *Infinite square well*
Given be a one-dimensional square well with infinite high walls, i.e.,

$$V(x) = \begin{cases} 0 & \text{for } 0 \le x \le L \\ +\infty & \text{else.}\end{cases}$$

The stationary wave function $\phi_n(x)$ and the associate energy eigenvalues E_n were calculated in the lecture.

(a) Show that the general time depending solution of the Schrödinger equation with the initial condition $\psi(x,0) = \psi_0(x)$ can be written in the form

$$\psi(x,t) = \sum_n \alpha_n \phi_n(x)\mathrm{e}^{-\frac{i}{\hbar}E_n t}.$$

How are the α_n determined?

(b) At time $t = 0$ let the wave function be $\psi(x, t = 0) = \psi_0(x)$. Let the period of the fundamental oscillation be T. Show that after a time $t = T/2$ the function passes into its reflection with respect to $x = L/2$, i.e. that $\psi(x, T/2) = -\psi(L - x, 0) = -\psi_0(L - x)$. What does happen after multiples of T?

Please turn over!

Problem 16. *Free fall II*
Let us reconsider Problem 13.

(a) Show that the stationary wave function of energy E is given by

$$\phi_E(x) = \mathcal{N} \mathrm{Ai}\left(\frac{x}{l_0} - \frac{E}{\epsilon_0}\right).$$

Here $\mathcal{N}$ is a normalization constant, $l_0 = (\hbar^2/2m^2g)^{1/3}$ and $\epsilon_0 = (\hbar^2 mg^2/2)^{1/3}$ are the standard length and energy, respectively.

$$\mathrm{Ai}(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \exp\left(i(yx + \frac{1}{3}y^3)\right)$$

is the Airy function. First show that this function suffices the differential equation

$$\mathrm{Ai}''(x) - x\mathrm{Ai}(x) = 0.$$

Its lowest zeros are $-2.3381\ldots$, $-4.0879\ldots$, $-5.52055\ldots$.

(b) Now the particle falls onto a hard surface at $x = 0$, meaning the potential be

$$V(x) = \begin{cases} mgx & \text{for } x > 0 \\ +\infty & \text{for } x \leq 0. \end{cases}$$

What has to hold for $\phi(x)$ when $x \leq 0$? What follows for the allowed energy values, if you use the above ansatz for the wave function in the case of $x > 0$?