Note: Please insert your assignment into the mailboxes 5th floor, building 46.

Problem 10. *Operator derivatives*
The derivative of a continuous operator $A(\lambda), \lambda \in \mathbb{R}$, is defined as

$$\frac{\mathrm{d}A(\lambda)}{\mathrm{d}\lambda} := \lim_{\epsilon \to 0} \frac{A(\lambda+\epsilon) - A(\lambda)}{\epsilon} \tag{1}$$

(provided the limit exist). Show:

(a)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(AB) = \frac{\mathrm{d}A}{\mathrm{d}\lambda}B + A\frac{\mathrm{d}B}{\mathrm{d}\lambda}; \tag{2}$$

(b) *

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}A^{-1} = -A^{-1}\frac{\mathrm{d}A}{\mathrm{d}\lambda}A^{-1}; \tag{3}$$

(c)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{e}^{\lambda C} = C\mathrm{e}^{\lambda C}, \quad \text{if} \quad C \neq C(\lambda); \tag{4}$$

(d)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}A^n = \sum_{l=1}^{n} A^{l-1}\frac{\mathrm{d}A}{\mathrm{d}\lambda}A^{n-l}, \quad n \in \mathbb{N}. \tag{5}$$

Determine especially for $A \neq A(\lambda), B \neq B(\lambda)$ the derivatives

$$\textbf{(e)} \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathrm{e}^{\lambda B}A\mathrm{e}^{-\lambda B}) \quad \text{and} \quad \textbf{(f)} \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathrm{e}^{\lambda A}\mathrm{e}^{\lambda B}). \tag{6}$$

*(Instruction to (b) : Use $A^{-1}A = \mathbb{1}$ and the product rule from (a).)

Problem 11. *Matrix exponential*
Given a selfadjoint matrix

$$\hat{A} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{7}$$

Determine the matrix exponential

$$\hat{T}(\alpha) := \mathrm{e}^{i\alpha\hat{A}}, \quad \alpha \in \mathbb{R}. \tag{8}$$

(a) Use the matrix exponential series (Taylor series).

(b) Use the spectral decomposition.

Problem 12. *Uncertainty principle*

(a) Let $[\hat{B},\hat{C}] = \hat{A}$ and $[\hat{A},\hat{C}] = \hat{B}$ be two operators. Show that

$$\Delta(\hat{A}\hat{B})\Delta\hat{C} \geq \frac{1}{2}\langle\hat{A}^2 + \hat{B}^2\rangle. \tag{9}$$

(b) In one spatial dimension holds:

$$\langle\Delta\hat{r}_\alpha^2\rangle\langle\Delta\hat{p}_\alpha^2\rangle \geq \frac{\hbar^2}{4}, \tag{10}$$

where $\alpha = x, y, z$. Show:

$$\langle\Delta\hat{\vec{r}}^{\,2}\rangle\langle\Delta\hat{\vec{p}}^{\,2}\rangle \geq \frac{9\hbar^2}{4}. \tag{11}$$

Use the simplifying assumption that $\langle\hat{r}_\alpha\rangle = \langle\hat{p}_\alpha\rangle = 0$ and use the inequality.

$$\frac{a}{b} + \frac{b}{a} \geq 2 \qquad \text{für } a, b > 0. \tag{12}$$

Problem 13. *Free fall*
The one dimensional Schrödinger equation in the presence of a constant force $F = -mg$ (free fall) can be written as:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + mgx\right]\psi(x,t). \tag{13}$$

(a) Derive from the above equation a partial differential equation of first order for the wave function $\psi(\tilde{k},t)$ in k-space.

(b) Show that the partial differential equation can be written as an ordinary differential equation, when substituting $k = k_0 + \frac{mg}{\hbar}t$

$$\frac{\partial}{\partial t}\tilde{\psi}(k_0 + \frac{mg}{\hbar^2}t, t) = -i\frac{\hbar}{2m}(k_0 + \frac{mg}{\hbar}t)^2\tilde{\psi}(k_0 + \frac{mg}{\hbar}t, t). \tag{14}$$

(c) Determine $\tilde{\psi}(k,t)$ and show that the average momentum $\langle\hat{p}\rangle = \hbar\langle k\rangle$ of a particle is a linear function in time t.