"Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the 'old one.' I, at any rate, am convinced that He does not throw dice." – Albert Einstein

Problem 6. *Fourier transformation*
Sometimes it is a good idea to consider the wave function not in the position space but the Fourier transformed wave function in the momentum space (k-space). In one dimension it holds:

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} \mathrm{d}x \; \mathrm{e}^{-ikx} \psi(x). \tag{1}$$

Show that for the expectation values of x^n and k^n holds:

$$\langle x^n \rangle = \int_{-\infty}^{\infty} \mathrm{d}x \; x^n |\psi(x)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \; \tilde{\psi}^*(k) \left(i \frac{\partial}{\partial k} \right)^n \tilde{\psi}(k), \tag{2}$$

$$\langle k^n \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \; k^n |\tilde{\psi}(k)|^2 = \int_{-\infty}^{\infty} \mathrm{d}x \; \psi^*(x) \left(i \frac{\partial}{\partial x} \right)^n \psi(x). \tag{3}$$

Problem 7. *Scalar product and expectation values*

(a) Show that the scalar product of two vectors $|f\rangle$ and $|g\rangle$ can be written as an integral in position space or momentum space respectively

$$\int \mathrm{d}x \; f^*(x) g(x) \qquad \text{or} \qquad \int \mathrm{d}k \; \tilde{f}^*(k) \tilde{g}(k). \tag{4}$$

(b) Show that since $\langle f|f\rangle = 1$ it follows

$$\int \mathrm{d}x \; |f(x)|^2 = 1 \qquad \text{or} \qquad \int \mathrm{d}k \; |\tilde{f}(k)|^2 = 1. \tag{5}$$

(c) Consider the Gaussian wave function

$$\Psi(x) = \mathcal{N} \mathrm{e}^{-x^2/d^2}. \tag{6}$$

Compute the normalization factor $\mathcal{N}$ and the expectation values

$$(i) \quad \langle \hat{x} \rangle = \langle \Psi | \hat{x} | \Psi \rangle \tag{7}$$

$$(ii) \quad \langle \hat{x}^2 \rangle = \langle \Psi | \hat{x}^2 | \Psi \rangle \tag{8}$$

$$(iii) \quad \langle \hat{p} \rangle = \langle \Psi | \hat{p} | \Psi \rangle \tag{9}$$

$$(iv) \quad \langle \hat{p}^2 \rangle = \langle \Psi | \hat{p}^2 | \Psi \rangle , \tag{10}$$

where $\hat{p} = -i\hbar\partial_x$.

Problem 8. *Matrices*

(a) Show that the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{11}$$

is non-hermitian, but has real eigenvalues. Further show that there is *no complete set* of eigenvectors.

(b) Given

$$R = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} \qquad \text{and} \qquad |\Psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \tag{12}$$

where $|a|^2 + |b|^2 = 1$. Determine $\langle\Psi| R^2 |\Psi\rangle$ in two different ways: (i) Compute $\langle R^2\rangle$ directly. (ii) Compute the eigenvalues and eigenvectors of R

$$R\,|r_n\rangle = r_n\,|r_n\rangle\,, \qquad n = 1, 2 \tag{13}$$

and write $|\Psi\rangle$ as a linear combination

$$|\Psi\rangle = c_1\,|r_1\rangle + c_2\,|r_2\rangle\,. \tag{14}$$

Then determine

$$\langle R^2\rangle = r_1^2|c_1|^2 + r_2^2|c_2|^2. \tag{15}$$

Problem 9. *Commutators*

(a) Let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be different operators. Show:

(i) $[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$,

(ii) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$.

(b) Let $\hat{A}$, $\hat{B}$ be hermitian operators. Show that $\hat{A}\hat{B}$ is hermitian, if $[\hat{A}, \hat{B}] = 0$.

(c) Let $\hat{A}$, $\hat{B}$ be two commuting operators and let $|a\rangle$ be the eigenstate of $\hat{A}$ with eigenvalue a. Show that $\hat{B}\,|a\rangle$ is also an eigenstate of $\hat{A}$. Find the corresponding eigenvalue to that state.