Note: Please insert your assignment into the mailboxes 5th floor, building 46. The problems marked by a ☗ have to be submitted.

☗ **Problem 36.** *Kinetic momentum*

Different from the canonical momentum $\hat{\vec{p}}$ the kinetic momentum $m\hat{\vec{v}} = \hat{\vec{p}} - \frac{q}{c}\vec{A}$ describes a charged particle in an external electromagnetic field and is invariant under gauge transformation. Show that for $\hat{\vec{v}}$ the following commutation relations hold:

$$[\hat{x}_i, m\hat{v}_j] = i\hbar\delta_{ij} \tag{1}$$

$$[m\hat{v}_i, m\hat{v}_j] = i\hbar\frac{q}{c}\epsilon_{ijk}B_k, \tag{2}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$.

☗ **Problem 37.** *Unitary invariants*

Let $\hat{A}, \hat{B}, \hat{C}$ be operators on a separable Hilbert space $\mathcal{H}$ and let $\hat{U}$ be a unitary operator on $\mathcal{H}$. Show:

(a)

$$\mathrm{Tr}\{\hat{A}\hat{B}\hat{C}\} = \mathrm{Tr}\{\hat{B}\hat{C}\hat{A}\} = \mathrm{Tr}\{\hat{C}\hat{A}\hat{B}\}. \tag{3}$$

(b)

$$\mathrm{Tr}\{\hat{A}\} = \mathrm{Tr}\{\hat{U}^{-1}\hat{A}\hat{U}\} = \sum_{i=1}^{\infty} a_i, \tag{4}$$

$$\det\{\hat{A}\} = \det\{\hat{U}^{-1}\hat{A}\hat{U}\} = \prod_{i=1}^{\infty} a_i, \tag{5}$$

where a_i are the eigenvalues of $\hat{A}$.

☗ **Problem 38.** *Determinant and trace 1*

Let $\hat{A}$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Show that for $\epsilon \ll 1$:

$$\det\{\mathbb{1} + \epsilon\hat{A}\} = 1 + \epsilon\mathrm{Tr}\{\hat{A}\} + \mathcal{O}(\epsilon^2). \tag{6}$$

Problem 39. *Determinant and trace 2*

Use problem 38 to show that for a self-adjoint operator on a separable Hilbert space holds:

$$\det\{\exp\{\hat{A}\}\} = \exp\{\mathrm{Tr}\{\hat{A}\}\}. \tag{7}$$

Hint: Consider $\hat{D}(\lambda) = \det\{\exp\{\lambda\hat{A}\}\}$ and write $\frac{\mathrm{d}\hat{D}(\lambda)}{\mathrm{d}\lambda}$ as the difference quotient. Alternatively consider a unitary transformation $\hat{U}$, that diagonalises $\hat{A}$.

Problem 40. *Landau levels - alternative access*

Let $\vec{B} = (0, 0, B)$ be a constant, homogeneous magnetic filed in z-direction. Let the electric field and the scalar potential be zero, so that the vector potential $\vec{A}$ is a function of space only. Show that the Hamilton operator

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - q\vec{A}\right)^2 = \hat{H}_\perp + \hat{H}_\parallel \tag{8}$$

can be separated in a transversal and longitudinal part

$$\hat{H}_\perp = \frac{m}{2}\left(\hat{v}_x^2 + \hat{v}_y^2\right), \qquad \hat{H}_\parallel = \frac{m}{2}\hat{v}_z^2. \tag{9}$$

Show that both parts commute, i.e. $[\hat{H}_\perp, \hat{H}_\parallel] = 0$. Define the following operators

$$\hat{Q} = \frac{\sqrt{m/M}}{\omega_c}\hat{v}_x, \qquad \hat{P} = \sqrt{mM}\hat{v}_y, \tag{10}$$

where $\omega_c = qB/m$ is the cyclotron frequency. Show that $\hat{Q}$ and $\hat{P}$ obey $[\hat{Q}, \hat{P}] = i\hbar$. Express $\hat{H}_\perp$ through $\hat{Q}$ and $\hat{P}$ and determine the eigenvalues of $\hat{H}_\perp$. What are the energy eigenvalues of $\hat{H}$ (*Landau levels*)?