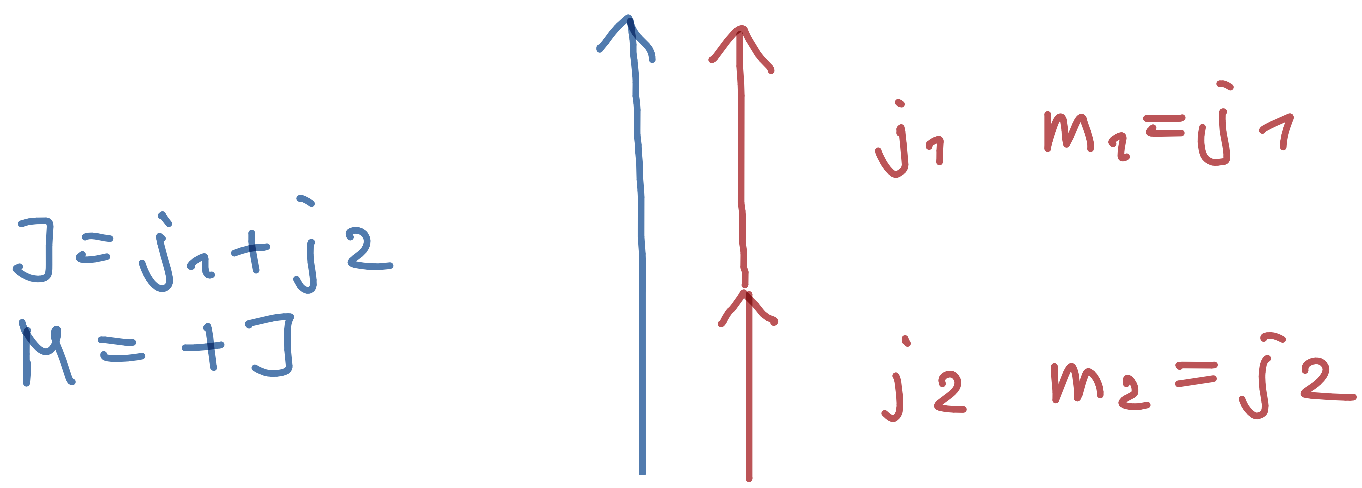


successive application of ladder operators $\hat{J}_\pm$ yields other M-states for $J = J_{max} = j_1 + j_2$

(447) $$\hat{J}_\pm |J, M\rangle = \sqrt{(J \pm M + 1)(J \mp M)}\, |J, M \pm 1\rangle$$

(448) using $$\hat{J}_\pm = \hat{J}_{1\pm} + \hat{J}_{2\pm}$$

one can easily construct the action on the uncoupled basis $|j_1 m_1 j_2 m_2\rangle$

so starting with

(449) $$\boxed{|J = j_1 + j_2, M = J\rangle = |j_1 j_1 j_2 j_2\rangle}$$

$$|J = j_1 + j_2, M = J - 1\rangle = \left(2(j_1 + j_2)\right)^{-1/2} \hat{J}_- |J = j_1 + j_2, M = J\rangle$$

(450) $$= \left(2(j_1 + j_2)\right)^{-1/2} \left(\hat{J}_{1-} + \hat{J}_{2-}\right) |j_1 j_1 j_2 j_2\rangle$$

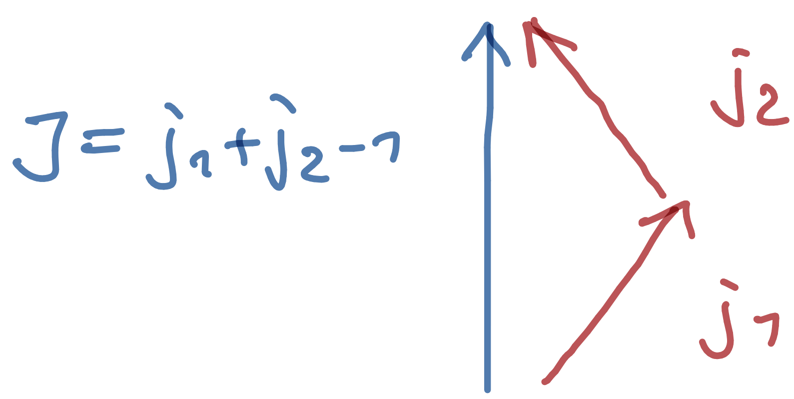
$$= \sqrt{\frac{j_1}{j_1 + j_2}}\, |j_1, j_1 - 1; j_2, j_2\rangle + \sqrt{\frac{j_2}{j_1 + j_2}}\, |j_1, j_1; j_2, j_2 - 1\rangle$$

$$|J=j_1+j_2, M=J-1\rangle =$$
$$(451)\quad \boxed{= \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1 j_1-1; j_2, j_2\rangle + \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1 j_1; j_2, j_2-1\rangle}$$

applying $\vec{J}_-$ again yields all states

$$|J=j_1+j_2; M=-(j_1+j_2), \dots, +(j_1+j_2)\rangle$$

(b) states with smaller $J < j_1+j_2$



consider orthogonal complement to $V_{J=j_1+j_2}$

next largest eigenvalue $M=j_1+j_2-1$
$\Rightarrow$ states

$$|j_1, j_1-1; j_2, j_2\rangle \qquad |j_1, j_1, j_2, j_2-1\rangle$$

find linear combination of these states that are orthogonal to (451)

$$|\psi\rangle = -\sqrt{\frac{j_2}{j_1+j_2}}\,|j_1,j_1-1;j_2,j_2\rangle + \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1 j_1; j_2, j_2-1\rangle \tag{452}$$?

check

$$\hat{J}_+|\psi\rangle = \hat{J}_{1+}|\psi\rangle + \hat{J}_{2+}|\psi\rangle$$

$$= -\sqrt{\frac{j_2}{j_1+j_2}}\sqrt{(j_1+j_1)(j_1-j_1+1)}\,|j_1 j_1 j_2 j_2\rangle \tag{453}$$

$$+ \sqrt{\frac{j_1}{j_1+j_2}}\sqrt{(j_2+j_2)(j_2-j_2+1)}\,|j_1 j_1 j_2 j_2\rangle = 0$$ ✓

$\Rightarrow$ maximal M state! does this correspond to $J = j_1+j_2-1$?

$$\hat{J}_z|\psi\rangle = (\hat{J}_{1z}+\hat{J}_{2z})|\psi\rangle = (j_1+j_2-1)|\psi\rangle$$

$$\Rightarrow \quad J = j_1+j_2-1 \quad \text{and} \quad M = J \tag{454}$$

so

$$\boxed{\begin{aligned}&|J=j_1+j_2-1;\,M=J\rangle = \\ &= -\sqrt{\frac{j_2}{j_1+j_2}}\,|j_1 j_1-1; j_2 j_2\rangle + \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1 j_1; j_2, j_2-1\rangle\end{aligned}} \tag{455}$$

Now again application of $\hat{J}_-$ generates all eigenstates

$$|J = j_1 + j_2 - 1\, ; M = -(j_1+j_2-1), \dots, (j_1+j_2-1)\rangle$$

One can continue and construct the states for $J = j_1 + j_2 - 2$, which becomes quickly tedious

in general

$$|J, M, j_1, j_2\rangle = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | J M\rangle \, |j_1 m_1 j_2 m_2\rangle$$

$$|j_1 m_1 j_2 m_2\rangle = \sum_{J=|j_1-j_2|}^{|j_1+j_2|} \sum_{M} \langle J M | j_1 m_1 j_2 m_2\rangle \, |J M\rangle$$

(456)

$\langle j_1 m_1 j_2 m_2 | J M\rangle$ Clebsch-Gordon coefficients

can always be chosen real

Rem.: $\exists$ explicit analytic expressions for the Clebsch Gordon coefficients but they are not very transparent

Some properties of Clebsch-Gordon coefficients

The Clebsch-Gordon coefficients can be summarized in a

$$(2j_1+1)(2j_2+1) \times (2j_1+1)(2j_2+1)$$

matrix

$$\underline{\underline{C}}(m_1 m_2 J M) \qquad \text{for given } j_1 j_2$$

$\underline{\underline{C}}$ is block diagonal where each block is characterized by $M = m_1 + m_2$

$M = M_{max} = j_1 + j_2$

only $J = j_1 + j_2 \qquad M = j_1 + j_2$
$m_1 = j_1 \qquad m_2 = j_2$

1×1 matrix

$M = M_{max} - 1 = j_1 + j_2 - 1$

$$\left.\begin{array}{l} J = j_1 + j_2 \\ J = j_1 + j_2 - 1 \end{array}\right\} \quad \left\{\begin{array}{ll} m_1 = j_1 & m_2 = j_2 - 1 \\ m_1 = j_1 - 1 & m_2 = j_2 \end{array}\right.$$

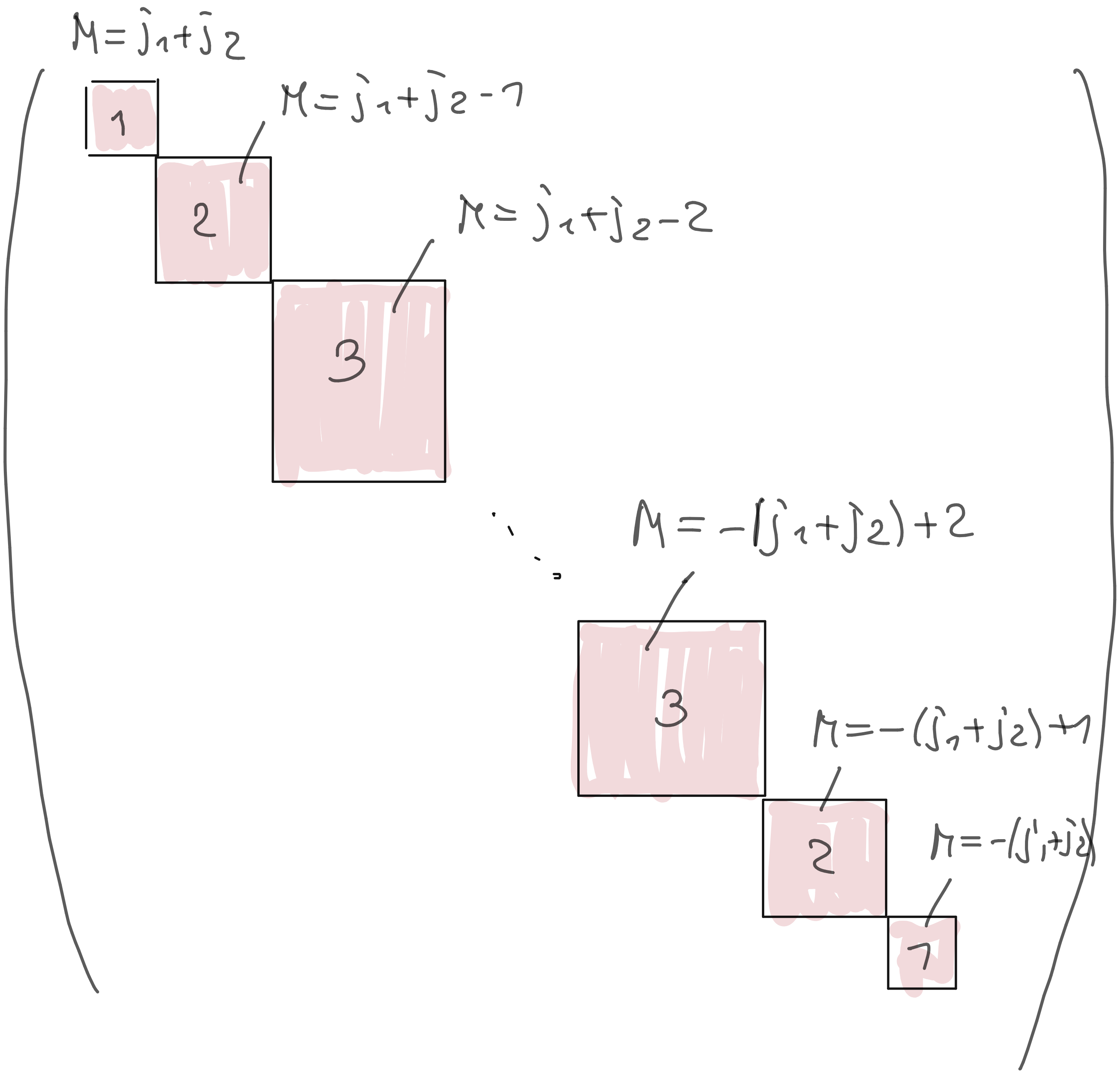
2×2 matrix

$$M = M_{max} - 2 = j_1 + j_2 - 2$$

$$\left.\begin{aligned} J &= j_1 + j_2 \\ J &= j_1 + j_2 - 1 \\ J &= j_1 + j_2 - 2 \end{aligned}\right\} \quad \left\{\begin{aligned} m_1 &= j_1 & m_2 &= j_2 - 2 \\ m_1 &= j_1 - 1 & m_2 &= j_2 - 1 \\ m_1 &= j_1 - 2 & m_2 &= j_2 \end{aligned}\right.$$

3×3 matrix

$\underline{\underline{C}} =$



(457)

recursion for Clebsh Gordon's

considering $\langle m_1 m_2 | \hat{J}_\pm | J M \rangle$ $\quad \hat{J}_\pm = \hat{J}_{1\pm} + \hat{J}_{2\pm}$

yields

(458)
$$\boxed{\begin{aligned} &\sqrt{J(J+1) - M(M+1)}\, \langle m_1 m_2 | J\, M+1 \rangle = \\ &= \sqrt{j_1(j_1+1) - m_1(m_1-1)}\, \langle m_1-1\, m_2 | J M \rangle \\ &+ \sqrt{j_2(j_2+1) - m_2(m_2-1)}\, \langle m_1, m_2-1 | J M \rangle \end{aligned}}$$

and

(459)
$$\boxed{\begin{aligned} &\sqrt{J(J+1) - M(M-1)}\, \langle m_1 m_2 | J\, M-1 \rangle = \\ &= \sqrt{j_1(j_1+1) - m_1(m_1+1)}\, \langle m_1+1\, m_2 | J M \rangle \\ &+ \sqrt{j_2(j_2+1) - m_2(m_2+1)}\, \langle m_1\, m_2+1 | J M \rangle \end{aligned}}$$

recursive construction of Clebsh Gordon matrix

$\underline{M = j_1 + j_2}$ $\quad$ trivial

$\underline{M = j_1 + j_2 - 1}$

- $J = j_1 + j_2$ recursion (459)

$$\langle m_1 m_2 | J M-1\rangle \longleftrightarrow \begin{array}{l} \langle m_1+1\, m_2 | J M\rangle \\ \langle m_1, m_2+1 | J M\rangle \end{array}$$

- $J = j_1 + j_2 - 1$

use basis $|m_1+1, m_2\rangle$, $|m_1, m_2+1\rangle$
and orthogonality; i.e.

$$\langle J = j_1 + j_2 ; M = j_1 + j_2 - 1 | J = j_1 + j_2 - 1 , M = j_1 + j_2 - 1\rangle$$
$$= 0$$

$M = j_1 + j_2 - 2$

- $J = j_1 + j_2$ recursion (459)

$$\langle m_1 m_2 | J M-2\rangle \longleftrightarrow \begin{array}{l} \langle m_1+1, m_2 | J M-1\rangle \\ \langle m_1, m_2+1 | J M-1\rangle \end{array}$$

- $J = j_1 + j_2 - 1$ recursion (459)

$$\langle m_1 m_2 | J = j_1 + j_2 - 1 , M = j_1 + j_2 - 2\rangle$$
$$\longleftrightarrow \begin{array}{l} \langle m_1+1, m_2 | J, M = j_1 + j_2 - 3\rangle \\ \langle m_1, m_2+1 | J, M = j_1 + j_2 - 3\rangle \end{array}$$

- $J = j_1 + j_2 - 2$ orthogonality etc.

(c) Matrix elements of scalar-, vector- and tensoroperators, Wigner Eckert theorem

▶ $\hat{S}$ scalar, i.e. rotationally invariant operator

(460) $$[\hat{\vec{J}}, \hat{S}] = 0$$

Thus the eigenbasis of $\hat{S}$ is

$$\{ |\alpha, J, M\rangle \}$$

$\Rightarrow$

(461) $$\langle \alpha J M | \hat{S} | \alpha' J' M' \rangle \sim \delta_{JJ'} \delta_{MM'}$$

Furthermore

$$\langle \alpha J M | \hat{S} | \alpha' J M \rangle = \sqrt{J(J+1) - M(M+1)} \, \langle \alpha J M | \underbrace{\hat{S} \hat{J}_+}_{\text{commute}} | \alpha' J M-1 \rangle$$

(462) $$= \sqrt{J(J+1) - M(M+1)} \, \langle \alpha J M | \hat{J}_+ \hat{S} | \alpha', J, M-1 \rangle$$

$$= \langle \alpha J M-1 | S | \alpha' J M-1 \rangle$$ i.e. independent on M !

(463) $$\boxed{\langle \alpha J M | \hat{S} | \alpha' J' M' \rangle = \delta_{JJ'} \delta_{MM'} S_{\alpha\alpha'}(J)}$$

only depends on α, α', J

▶ Generalization to vector- or tensor operators

Def: A tensor $T_{ijk\ldots}$ is called irreducible if the space in which is defined is irreducible under rotations, i.e. cannot be decomposed into invariant subspaces

example: vectors in $\mathbb{R}^3$, spinors

Def: The $2j+1$ operators $\hat{T}_m^{(j)}$ with $m = -j, -j+1, \ldots, j-1, j$ are called standard components of an irreducible tensor operator if they transform under rotations

$$|j,m\rangle \longrightarrow R|jm\rangle = \sum_{m'} R^{(j)}_{mm'} |jm'\rangle \tag{464}$$

basis states of angular momentum

as

$$\hat{T}_m^{(j)} \longrightarrow R\hat{T}_m^{(j)} R^{-1} = \sum_{m'} R^{(j)}_{mm'} \hat{T}^{(j)}_{m'} \tag{465}$$

This is equivalent to ($\hbar = 1$)

$$\boxed{\begin{aligned} \left[\hat{J}_\pm, \hat{T}_m^{(j)}\right] &= \sqrt{j(j+1) - m(m\pm 1)}\ \hat{T}^{(j)}_{m\pm 1} \\ \left[\hat{J}_z, \hat{T}_m^{(j)}\right] &= m\,\hat{T}_m^{(j)} \end{aligned}} \tag{466}$$

Wigner-Eckert theorem

The matrix elements of irreducible tensor operators $\hat{T}^{(j)}_m$ have the following structure

$$\langle \alpha J M | \hat{T}^{(j)}_m | \alpha' J' M' \rangle =$$

$$= \frac{1}{\sqrt{2J+1}} \underbrace{\langle \alpha J \| \hat{T}^{(j)}_m \| \alpha' J' \rangle}_{\text{only depends on } \alpha, \alpha', J, J'} \langle J' M' j m | J M \rangle \tag{467}$$

An important application of the Wigner-Eckart theorem are selection rules

(468)
$$\boxed{\begin{gathered}\langle \alpha, J, M | \hat{T}^{(j)}_m | \alpha', J', M' \rangle \neq 0 \\ \text{if and only if} \\ m = M - M' \quad \text{and} \quad |J - J'| \leq j \leq J + J'\end{gathered}}$$

END