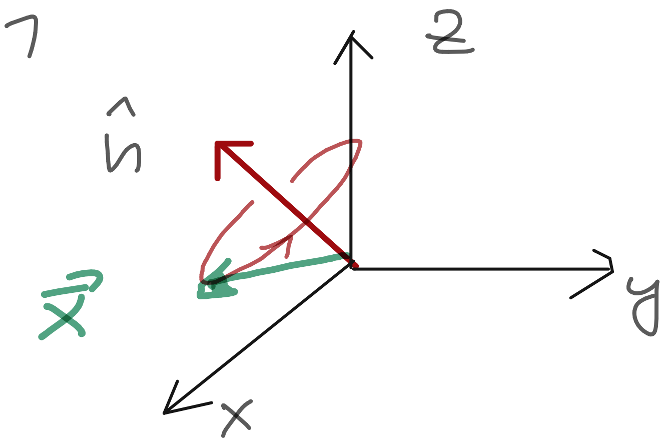
# VIII.4 Rotations

group of rotations in $\mathbb{R}^3$: orthogonal matrices $R$

(399) $$\det(R) = +1 \qquad SO(3)$$

rotation axis $\hat{n}$ $\qquad |\hat{n}| = 1$

rotation angle $\varphi$




$$\vec{y} = R(\hat{n}, \varphi)\,\vec{x}$$

Infinitesimal rotation

(400) $$R(\hat{n}, \delta\phi)\,\vec{x} = \vec{x} + \delta\phi\,(\hat{n} \times \vec{x}) + \mathcal{O}(\delta\phi^2)$$

introduce hermitian matrices $\ell_j$

(401) $$\boxed{(\ell_j)_{km} = i\,\varepsilon_{jkm}}$$

then

$$R(\hat{n}, \delta\phi) = \mathbb{1} - i\,\delta\phi\,\hat{n}_j \ell_j$$

finite rotation $\delta\phi = \phi/N \qquad N \to \infty$

$$R(\hat{n}, \phi) = \left(\mathbb{1} - i\frac{\phi}{N}\hat{n}_j \ell_j\right)^N$$

(402)

$$= \exp(-i\varphi\,\hat{n}_j \ell_j)$$

## (A) Unitary representation of rotation group for spin-less particle

unitary operation for transformation of wavefunction under rotation

$$\vec{r} \xleftarrow{\;R(\hat{n},\varphi)\;} \longrightarrow \vec{r}\,' = R(\hat{n},\varphi)\vec{r}$$

A B

(403) $$\psi_A(\vec{r}) \stackrel{!}{=} \psi_B(\vec{r}\,') = \psi_B(R\vec{r})$$

(404)
$$\psi_B(\vec{r}) = U_R(\varphi,\vec{n})\,\psi_A(\vec{r})$$
$$= \psi_A(R^{-1}(\hat{n},\varphi)\,\vec{r})$$

Similar to previous related discussion:
infinitesimal rotation

$$\psi_A(R^{-1}(\hat{n}\,\delta\varphi)\vec{r}) = \psi_A((\mathbb{1} + i\delta\phi\, n_j\, \ell_j)\vec{r})$$

(405) $\Rightarrow$ $$U_R(\delta\varphi,\vec{n}) = e^{-i\delta\phi\frac{\hat{n}\cdot\vec{L}}{\hbar}}$$

finite trafo

(406) $$\boxed{U_R(\phi,\vec{n}) = e^{-i\phi\frac{\hat{n}\cdot\vec{L}}{\hbar}}}$$

Lie group of unitary rotation operators

generators: $\hat{L}_\mu$

(407) $$[\hat{L}_i, \hat{L}_j] = i\varepsilon_{ijk}\hat{L}_k \qquad (\hbar = 1)$$

Casimir operator $\hat{L}^2$
(408)

## (B) Unitary representation of rotation group for spin 1/2 particle

Let $$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$ a 2-component spinor

translation

Does not affect internal structure, i.e.

$\vec{r}$ $\qquad\qquad$ $\vec{r}\,' = \vec{r} + \vec{a}$

A $\qquad\qquad$ B

$$\psi_A(\vec{r}) \stackrel{!}{=} \psi_B(\vec{r}\,') = \psi_B(\vec{r}+\vec{a})$$

i.e.

(408) $$\psi_B(\vec{r}) = \psi_A(\vec{r}-\vec{a}) = e^{-\frac{i\vec{a}\cdot\hat{\vec{p}}}{\hbar}}\psi_A(\vec{r})$$

## how about rotation

(409) $$\psi'(\vec{r},t) = \hat{U}_{R(\hat{n},\phi)}\,\psi(\vec{r},t)$$

$\hat{U}_{R(\hat{n},\phi)}$ in general $2\times 2$ matrix in spinor space

In order to determine $\hat{U}_{R(\hat{n},\phi)}$ we require

(i) $\psi^\dagger\psi$ behaves like a scalar upon rotation

(ii) $\vec{\hat{S}} = \frac{\hbar}{2}\vec{\sigma}$ behaves like a vector upon rotation

from (i) $\Rightarrow$

$$\psi'^\dagger(\vec{r})\,\psi'(\vec{r}) = \psi^\dagger(R^{-1}\vec{r})\,\psi(R^{-1}\vec{r})$$

$$= \sum_{m=1}^{2} \psi_m^*(R^{-1}\vec{r})\,\psi_m(R^{-1}\vec{r})$$

(410) $$= \sum_{m=1}^{2} \left[e^{-\frac{i}{\hbar}\vec{\phi}\cdot\vec{\hat{L}}}\psi_m(\vec{r})\right]^*\left[e^{-\frac{i}{\hbar}\vec{\phi}\cdot\vec{\hat{L}}}\psi_m(\vec{r})\right]$$

$$= \left[U_L\,\psi(\vec{r})\right]^\dagger\left[U_L\,\psi(\vec{r})\right]$$

where

(411) $$U_L = e^{-\frac{i}{\hbar}\vec{\phi}\cdot\vec{\hat{L}}} = e^{-\frac{i}{\hbar}\phi\,\vec{n}\cdot\vec{\hat{L}}}$$

is the usual rotation operator

$\Rightarrow$ <u>ansatz</u>

(412) $$\boxed{U_R = U_S U_L}$$

where

(413) $$U_S = \exp\left\{-\frac{i}{\hbar}\varphi\,\hat{n}\cdot\vec{a}\right\}$$

and we still have to determine the vectoroperator $\vec{a}$

from (ii)

(414) $$U^{+}_{S(\varphi,\hat{n})}\,\sigma_i\,U_{S(\varphi,\hat{n})} = R^{-1}_{ij}\,\sigma_j$$
$$= R_{ji}\,\sigma_j$$

- infinitesimal rotation

$$\vec{r}\,' = \vec{r} + \delta\phi\,\hat{n}\times\vec{r}$$

$$x_i' = x_i + \delta\phi\,\varepsilon_{ijk}\,n_j\,x_k = R_{ik}\,x_k$$

i.e.

(415) $$R_{ik}(\hat{n},\delta\phi) = \delta_{ik} + \varepsilon_{ijk}\,n_j\,\delta\phi$$

using (414) i.e.

(416) $$R_{ij}\,U^{+}_{S}\,\sigma_i\,U_S = \sigma_j$$

and

(417) $$U_S(\hat{n},\delta\phi) = 1 - \frac{i}{\hbar}\delta\phi\,\hat{n}\cdot\vec{a}$$

we find

$$\hat{\sigma}_j = \left(\delta_{ij} + \varepsilon_{ikj} n_k \delta\phi\right)\left(1 + \frac{i}{\hbar}\delta\phi\, \vec{n}\cdot\vec{\hat{a}}\right)\cdot\hat{\sigma}_i \cdot \left(1 - \frac{i}{\hbar}\delta\phi\, \vec{n}\cdot\vec{\hat{a}}\right)$$

$$= \sigma_j + \varepsilon_{ikj}\,\delta\phi\, n_k \hat{\sigma}_i + \frac{i}{\hbar}\delta\phi\, n_k [\hat{a}_k, \hat{\sigma}_j] + \mathcal{O}(\delta\phi^2)$$

$\Rightarrow$

(418) $$[\hat{a}_k, \hat{\sigma}_j] \overset{!}{=} i\hbar\,\varepsilon_{ikj}\,\sigma_i$$

Since $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ are a complete set of $2\times 2$ matrices it follows

(419) $$\hat{a}_k = \alpha_{km}\hat{\sigma}_m + \beta_k \mathbb{1}$$

Since $\beta_k \mathbb{1}$ only gives an overall $U(1)$ phase which can always be added we can set $\beta_k = 0$ and we find

(420) $$\alpha_{km} = \frac{\hbar}{2}\delta_{km}$$

Thus

(421) $$\boxed{\hat{U}_{S(\vec{n},\phi)} = \exp\left\{-\frac{i}{2}\phi\, \vec{n}\cdot\vec{\hat{\sigma}}\right\}}$$

and we find for the total rotation operator

$$\boxed{\hat{U}_{R(\vec{n},\phi)} = \hat{U}_S \hat{U}_L = \exp\left\{-\frac{i}{\hbar}\phi\,\vec{n}\cdot\left(\frac{\hbar}{2}\vec{\sigma}+\vec{L}\right)\right\} = \exp\left\{-\frac{i}{\hbar}\phi\,\vec{n}\cdot\vec{J}\right\}} \tag{422}$$

$$\boxed{\vec{J} = \vec{L} + \vec{S}} \tag{423}$$

## VIII.5 Algebra of angular momenta

### (A) General properties

Def: Every vector operator $\vec{J}$ whose components fulfill the Lie algebra SO(3)

$$\boxed{[\hat{J}_i, \hat{J}_j] = i\hbar\,\varepsilon_{ijk}\hat{J}_k} \tag{424}$$

is an angular momentum

Casimir operator $\quad \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$

$$[\hat{J}_i, \hat{J}^2] = 0 \tag{425}$$

## Laddeoperators

(426) $$\hat{J}_\pm = \hat{J}_x \pm i \hat{J}_y$$

(427) $$[\hat{J}_z, \hat{J}_\pm] = \pm \hbar \hat{J}_\pm \qquad [\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$$

The Lie algebra fixes the eigenvalues and eigenstates ($\hbar = 1$)

(428) $$\hat{J}^2 = \frac{1}{2}\left(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\right) + \hat{J}_z^2$$

(429) $$\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z(\hat{J}_z - 1)$$

(430) $$\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z(\hat{J}_z + 1)$$

$\hookrightarrow$

(431) $$\boxed{\begin{aligned} \hat{J}^2 |j,m\rangle &= j(j+1)|j,m\rangle \\ \hat{J}_z |j,m\rangle &= m|j,m\rangle \end{aligned}}$$

One finds from (430) $\| \hat{J}_+ |jm\rangle \| = \langle jm | \hat{J}_- \hat{J}_+ | jm \rangle$

(432) $$\boxed{\hat{J}_+ |j,m\rangle = \sqrt{j(j+1) - m(m+1)}\, |j,m+1\rangle}$$

and from (429) $\| \hat{J}_- |jm\rangle \| = \langle jm | \hat{J}_+ \hat{J}_- | jm \rangle$

(433) $$\boxed{\hat{J}_- |j,m\rangle = \sqrt{j(j+1) - m(m-1)}\, |j,m-1\rangle}$$

## matrix representations of angular momentum operators

(434) $\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$ $\quad\quad$ $\hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)$

from (432), (433) $\Rightarrow$

$\boxed{j = \frac{1}{2}}$

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_x$$

$$J_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\sigma_y \qquad \mathbb{C}^2$$

$$J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_z$$

$\boxed{j = 1}$

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(436)
$$J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \mathbb{C}^3$$

$$J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\boxed{j = \frac{3}{2}}$

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{2} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$J_y = \frac{1}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix} \qquad \mathbb{C}^4 \tag{437}$$

$$J_z = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Thus we see that the unitary operator of rotation can be decomposed into block matrices acting on the multiplets

$$U = e^{-i\phi\hat{n}\cdot\vec{J}} = \left(\begin{array}{c|c|c|c} \overset{j=0}{U_1} & & & \\ \hline & \overset{j=1/2}{U_2} & & \\ \hline & & \overset{j=1}{U_3} & \cdots \\ \hline & & \vdots & \ddots \end{array}\right)$$

The above listed representations of $\hat{U}(\vec{n},\phi)$ on the multiplets are irreducible

# (B) Addition of angular momenta

If $\vec{\hat{J}}_1$ and $\vec{\hat{J}}_2$ are angular momenta then

$$\vec{\hat{J}} = \vec{\hat{J}}_1 + \vec{\hat{J}}_2 \tag{438}$$

is also an angular momentum if $[\vec{\hat{J}}_1, \vec{\hat{J}}_2] = 0$

<u>proof:</u>

$$[\hat{J}_i, \hat{J}_j] = [\hat{J}_{1i} + \hat{J}_{2i}, \hat{J}_{1j} + \hat{J}_{2j}]$$

$$= [\hat{J}_{1i}, \hat{J}_{1j}] + [\hat{J}_{2i}, \hat{J}_{2j}]$$

$$= i\varepsilon_{ijk}(\hat{J}_{1k} + \hat{J}_{2k}) = i\varepsilon_{ijk}\hat{J}_k \quad \checkmark$$

Let $\vec{\hat{J}}_1$ be an irreducible representation of the rotation group on $\mathbb{C}^{2j_1+1}$ and $\vec{\hat{J}}_2$ on $\mathbb{C}^{2j_2+1}$

Then $\vec{\hat{J}}_1 + \vec{\hat{J}}_2$ acts on $\mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j_2+1}$

A complete set of basis vectors is

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = |j_1 m_1 j_2 m_2\rangle \tag{439}$$

The dimension of $\mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j_2+1}$ is

(440) $$\dim\left(\mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j_2+1}\right) = (2j_1+1)(2j_2+1)$$

This space can be decomposed into multiplets (invariant subspaces) $V_J$ with

(441) $$\boxed{|j_1 - j_2| \leq J \leq j_1 + j_2}$$

and $\quad j_1 + j_2 - J \quad$ integer

i.e.

(442) $$\boxed{\mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j_2+1} = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} V_J}$$

where

(443) $$\dim(V_J) = 2J+1$$

proof: assume $j_1 \geq j_2$

$$\sum_{J=j_1-j_2}^{j_1+j_2} 2J+1 = \frac{(j_1+j_2)-(j_1-j_2)+1}{2}\left[2(j_1+j_2)+2(j_1-j_2)+2\right]$$

$$= (2j_2+1)(2j_1+1) \quad \square$$

Let us consider a multiplett with $J$

$\boxed{V_J}$ eigenvalue of Casimir operator $\hat{J}^2$

$J(J+1)$ $(\hbar = 1)$

set of commuting operators

$$\boxed{\{ \hat{J}^2, \hat{J}_z, \hat{J}_1^2, \hat{J}_2^2 \}} \tag{444}$$

- Expression of eigenstates $|J, M, j_1, j_2\rangle$ in terms of $|j_1 m_1, j_2 m_2\rangle$ ?

$$\hat{J}_z |j_1 m_1 j_2 m_2\rangle = (m_1 + m_2)|j_1 m_1 j_2 m_2\rangle$$

a) maximum eigenvalue

$M = m_1 + m_2 = j_1 + j_2$ must correspond to $J = j_1 + j_2$

$$\hat{J}_z |j_1 j_1 j_2 j_2\rangle = (j_1 + j_2)|j_1 j_1 j_2 j_2\rangle$$
$$\hat{J}^2 |j_1 j_1 j_2 j_2\rangle = (j_1 + j_2)(j_1 + j_2 + 1)|j_1 j_1 j_2 j_2\rangle \tag{445}$$

so

$$\boxed{|J = j_1 + j_2, M = j_1 + j_2, j_1, j_2\rangle = |j_1, j_1; j_2, j_2\rangle} \tag{446}$$

$\hat{J}^2, \hat{J}_z, \hat{J}_1^2, \hat{J}_2^2$ $\qquad$ $\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2, \hat{J}_{2z}$