(B) normal subgroup

group G of unitary operators $g_r^{+} = g_r^{-1}$
elements $\{g_r\}$
subgroup A elements $\{a_j\}$

Def: • A is called a normal subgroup of G if

$\forall g_r \in G$ and $\forall a_i \in A$

(390) $$g_r a_i g_r^{-1} = a_j$$

i.e. a unitary transformation of a_i stays in the subgroup A

Def: • A is called abelian normal subgroup if in addition

(391) $$a_i a_j = a_j a_i \quad \forall a_i, a_j \in A$$

example translation-rotation group
generators:

$$\hat{P}_x, \hat{P}_y, \hat{P}_z, \hat{L}_x, \hat{L}_y, \hat{L}_z$$

translation is an abelian normal subgroup as

$$RTR^{-1} = T'$$

where $T = e^{i\frac{a\hat{p}}{\hbar}}$ and $R = e^{i\frac{\theta \hat{L}}{\hbar}}$
are translations and rotations

<u>Def:</u> A Lie group is called simple if she does not have a continuous normal subgroup. It is called semi-simple if it does not contain an abelian, normal subgroup

The above definitions have pendents on the level of the Lie algebra

generators of G: $\quad \hat{G}_k$

generators of A: $\quad \hat{A}_k$

- For A to be a subgroup of G

(3.52) $$[\hat{A}_l, \hat{A}_m] = C_{lmn} \hat{A}_n$$

i.e. generators of A form a (sub) Lie algebra

- For A to be a normal subgroup of G we had

(3.53) $$g_r a_i g_r^{-1} = a_j \in A$$

i.e.

$$g_r a_i g_r^{-1} a_j^{-1} = a_j a_i^{-1} = a_m \in A$$

infinitesimal unitary operators

$$g_r = \mathbb{1} - i\,\delta\alpha_i^{(r)}\hat{G}_i - \frac{1}{2}\,\delta\alpha_i^{(r)}\delta\alpha_j^{(r)}\hat{G}_i\hat{G}_j$$

$$a_i = \mathbb{1} - i\,\delta\beta_k^{(i)}\hat{A}_k - \frac{1}{2}\,\delta\beta_k^{(i)}\delta\beta_e^{(i)}\hat{A}_k\hat{A}_e$$

Following similar steps as for the derivation of (387) we then find

$$\left[\hat{G}_i, \hat{A}_k\right] = a_{ikn}\hat{A}_n$$

<u>Def:</u> A sub algebra $\{\hat{A}_k\}$ of a Lie algebra $\{\hat{G}_e\}$ is called <u>ideal</u> if for $\forall\, \hat{G}_i, \hat{A}_k$ holds

$$\left[\hat{G}_i, \hat{A}_k\right] = a_{ikn}\hat{A}_n \tag{394}$$

<u>Def:</u> A Lie Algebra $\{\hat{G}_e\}$ is called <u>simple</u> if she does not contain an ideal except for $\{0\}$. A Lie Algebra is called <u>semi-simple</u> if it does not contain an abelian ideal

Cartan Let C_{ijk} be the structure constants of a Lie Algebra. The Lie Algebra is semi-simple if and only if

$$\mathrm{Det}(g_{\alpha\beta}) \neq 0 \tag{395}$$

$$g_{\alpha\beta} = g_{\beta\alpha} = C_{\alpha\mu\nu}\,C_{\beta\nu\mu}$$

example • $SO(3)$ special orthogonal group in $\mathbb{R}^3$

generators $\hat{L}_i$ $\quad [\hat{L}_i, \hat{L}_j] = i\varepsilon_{ijk}\hat{L}_k$

$$g_{\alpha\beta} = i\varepsilon_{\alpha\mu\nu}\, i\varepsilon_{\beta\nu\mu}$$
$$= \varepsilon_{\alpha\mu\nu}\varepsilon_{\beta\mu\nu} = 2\delta_{\alpha\beta}$$
$$\det g_{\alpha\beta} = 8 \neq 0$$

• translational group

generators $\hat{p}_i$ $\quad [\hat{p}_i, \hat{p}_j] = 0$

$$g_{\alpha\beta} = 0 \qquad \det g_{\alpha\beta} = 0$$

(c) Casimir operators

Racah For every semi-simple Lie group of rank ℓ there exists ℓ invariant operators (Casimir operators) $\hat{C}_\lambda$ $\lambda = 1, 2, \ldots, \ell$. They are functions of the generators and commute with all elements of the group

example $SO(3)$: semi-simple (see above)
rank 1

$\Rightarrow \exists$ 1 Casimir operator $\hat{L}^2$

(396) $$[\hat{L}^2, \hat{L}_i] = 0$$

If a Lie group is a symmetry group of a quantum mechanical system, then the eigenvalues of the Casimir operators define Multipletts. The eigenstates corresponding to the same eigenvalue are a basis of an invariant sub-space

example $\underbrace{Y_{00}(\vartheta,\varphi)}_{\text{1-dim supspace}}$; $\underbrace{Y_{11}(\vartheta,\varphi), Y_{10}(\vartheta,\varphi), Y_{1-1}(\vartheta,\varphi)}_{\text{3-dim subspace}}$

$$A = \{Y_{00}\} \oplus \{Y_{11}, Y_{10}, Y_{1-1}\}$$

The elements of the rotation group $(SO(3))$ $\hat{U}(\theta_\mu) = e^{-i\hat{L}_\mu \theta_\mu/\hbar}$ can be represented as matrix operations acting only on the invariant sub-spaces

matrix representation

$\ell=0$, $\ell=1$, $\ell=2$

$$\hat{U}(\theta_\mu) = \left(\begin{array}{c|ccc|c} U_0 & & & & \\ \hline & U_{-1-1} & U_{-10} & U_{-11} & \\ & U_{0-1} & U_{00} & U_{01} & \\ & U_{1-1} & U_{10} & U_{1-1} & \\ \hline & & & & \ddots \end{array}\right) \qquad (3.97)$$

<u>Def:</u> If the $\hat{U}(\alpha_\mu)$ form a Lie group of unitary operators in a Hilbert space $\mathcal{H}$, then the matrix representation of the group $\hat{U}(\alpha_\mu)$ is called **reducible** if invariant sub-spaces exist. Else $\hat{U}(\alpha)$ is called **irreducible**

$$\boxed{\text{The Hamiltonian is degenerate on the multipletts}}$$

<u>proof:</u> $|\psi_0\rangle$ eigenstate of multiplett

$$\hat{H}|\psi_0\rangle = E\,|\psi_0\rangle$$

$\hat{U}(\alpha)\,|\psi_0\rangle$ is again element of multiplett

$$\hat{H}\hat{U}\,|\psi_0\rangle = \hat{U}\hat{H}\,|\psi_0\rangle = E\,\hat{U}\,|\psi_0\rangle \qquad \square$$

remark: There is no general procedure to construct the Casimir operators

For SU(n) holds that the Casimir operators are simple polynomials of the generators

theorem	Every operator $\hat{A}$ which commutes with all elements of a Lie group (i.e. with all generators $\hat{I}_\mu$) is a function of the Casimir operators of the group

$\Rightarrow$

$$\boxed{\text{If a Lie group is a symmetry group the Hamiltonian is a function of the Casimir operators}}$$

example: • translational invariant system

rank 3 $\Rightarrow$ 3 Casimir operators ($\hat{p}_x, \hat{p}_y, \hat{p}_z$)

$$H = \alpha + \vec{\beta} \cdot \vec{p} + \gamma \vec{p}^{\,2} + \dots \tag{398}$$

• if there is time-reversal symmetry

$$THT^{-1} = H \qquad T\vec{p}T^{-1} = -\vec{p}$$

H can only contain even powers

$$H = \frac{\hat{p}^2}{2m} \quad \text{as opposed to} \quad H = \frac{1}{2m}\left(\vec{\hat{p}} - \frac{e}{c}\vec{A}\right)^2$$

TR invariant   not TR invariant

Some important groups

group	generators	rank	Casimir operator	type
Translation	$\hat{p}_x, \hat{p}_y, \hat{p}_z$	3	$\hat{p}_x, \hat{p}_y, \hat{p}_z$	abelian
rotation	$\hat{L}_x, \hat{L}_y, \hat{L}_z$	1	$\hat{L}^2 = \vec{\hat{L}} \cdot \vec{\hat{L}}$	(semi) simple
euklidian group transl. + rotation	$\hat{L}_x, \hat{L}_y, \hat{L}_z$ $\hat{p}_x, \hat{p}_y, \hat{p}_z$	3	$\hat{p}^2, \vec{\hat{p}} \cdot \vec{\hat{L}}$	not semi-simple no Racah theorem
Inversion	$\hat{\underline{P}}$	1	$\hat{\underline{P}}$	discrete (not Lie)

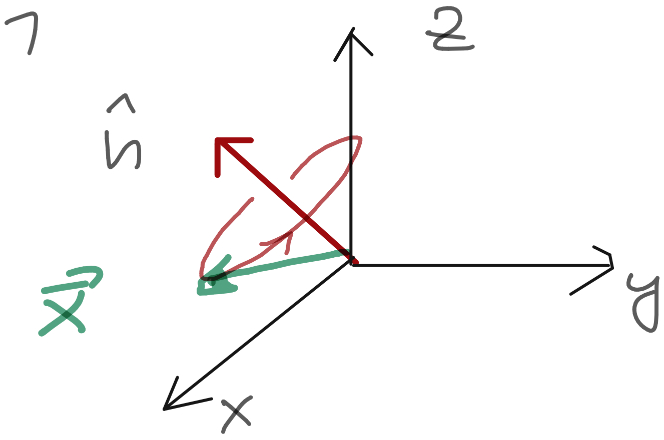
VIII.4 Rotations

group of rotations in $\mathbb{R}^3$: orthogonal matrices R

(399) $$\det(R) = +1 \qquad SO(3)$$

rotation axis $\hat{n}$ $\qquad |\hat{n}| = 1$

rotation angle φ



$$\vec{y} = R(\hat{n}, \varphi)\vec{x}$$

Infinitesimal rotation

(400) $$R(\hat{n}, \delta\phi)\vec{x} = \vec{x} + \delta\phi(\hat{n} \times \vec{x}) + \mathcal{O}(\delta\phi^2)$$

introduce hermitian matrices ℓ_j

(401) $$\boxed{(\ell_j)_{km} = i\,\varepsilon_{jkm}}$$

then

$$R(\hat{n}, \delta\phi) = \mathbb{1} - i\,\delta\phi\, \hat{n}_j \ell_j$$

finite rotation $\quad \delta\phi = \phi/N \qquad N \to \infty$

(402) $$R(\hat{n}, \phi) = \left(\mathbb{1} - i\frac{\phi}{N}\hat{n}_j \ell_j\right)^N$$

$$= \exp(-i\varphi\, \hat{n}_j \ell_j)$$

(A) Unitary representation of rotation group for spin-less particle

unitary operation for transformation of wavefunction under rotation

$$\vec{r} \xleftarrow{\quad} R(\hat{n},\varphi) \xrightarrow{\quad} \vec{r}\,' = R(\hat{n},\varphi)\,\vec{r}$$

A $\qquad\qquad\qquad\qquad$ B

(403) $$\psi_A(\vec{r}) \overset{!}{=} \psi_B(\vec{r}\,') = \psi_B(R\vec{r})$$

$$\psi_B(\vec{r}) = U_R(\varphi,\vec{n})\,\psi_A(\vec{r})$$

(404) $$= \psi_A(R^{-1}(\hat{n},\varphi)\,\vec{r})$$

Similar to previous related discussion: infinitesimal rotation

$$\psi_A(R^{-1}(\hat{n}\,\delta\varphi)\vec{r}) = \psi_A((\mathbb{1} + i\delta\phi\, n_j\, \ell_j)\vec{r})$$

(405) $\Rightarrow$ $$U_R(\delta\varphi,\vec{n}) = e^{-i\delta\phi\,\frac{\hat{n}\cdot\vec{L}}{\hbar}}$$

finite trafo

(406) $$\boxed{U_R(\phi,\vec{n}) = e^{-i\phi\,\frac{\hat{n}\cdot\vec{L}}{\hbar}}}$$