VIII Continuous symmetries in quantum mechanics

VIII.1 Continuous symmetries in classical mechanics

In classical mechanics there is a famous relation between continuous symmetries and constants of motion

Noether theorem:

If the Euler-Lagrange equations of a mechanical system are invariant under continuous coordinate transformations

$$(t;\vec{r}) \longrightarrow (t'(t);\vec{r}'(\vec{r},t))$$

then there exists a constant of motion associated with this symmetry.

(A) Homogeneity in space

(357) $$L(\vec{q}_i,\dot{\vec{q}}_i,t) \overset{!}{=} L(\vec{q}_i+\vec{a},\dot{\vec{q}}_i,t) \qquad \forall \vec{a}$$

$$\Rightarrow \sum_j \frac{\partial L}{\partial \vec{q}_j} = 0 = \sum_j \left(\frac{\partial L}{\partial x_j}, \frac{\partial L}{\partial y_j}, \frac{\partial L}{\partial z_j} \right)$$

where $x_j = q_{xj}$ etc.

Euler-Lagrange

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0$$

$$\Rightarrow \sum_j \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} = \frac{d}{dt} P_x = 0$$

conserved: total momentum

(358) $$\boxed{P_x = \text{const}}$$

Similarly

(359) $$\boxed{P_y = P_z = \text{const}}$$

If $\vec{a} = a\hat{n}$ with $\hat{n}$ being a unit vector and

$$L(\vec{q}_i, \dot{\vec{q}}_i, t) = L(\vec{q}_i + a\hat{n}, \dot{\vec{q}}_i, t) \quad \forall a$$

then only

(360) $$\boxed{P_n = \text{const}}$$

(B) Homogeneity in time

(361) $$L(\vec{q}_i, \dot{\vec{q}}_i, t) = L(\vec{q}_i, \dot{\vec{q}}_i)$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial \vec{q}_j} \dot{\vec{q}}_j + \sum_j \frac{\partial L}{\partial \dot{\vec{q}}_j} \ddot{\vec{q}}_j + \underbrace{\frac{\partial L}{\partial t}}_{=0}$$

Euler-Lagrange

$$= \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{q}}_j}\right) \dot{\vec{q}}_j + \sum_j \frac{\partial L}{\partial \dot{\vec{q}}_j} \ddot{\vec{q}}_j$$

$$= \frac{d}{dt} \sum_j \left(\frac{\partial L}{\partial \dot{\vec{q}}_j} \dot{\vec{q}}_j \right)$$

$\Rightarrow$

(362) $$\frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{\vec{q}}_j} \dot{\vec{q}}_j - L \right) = 0$$

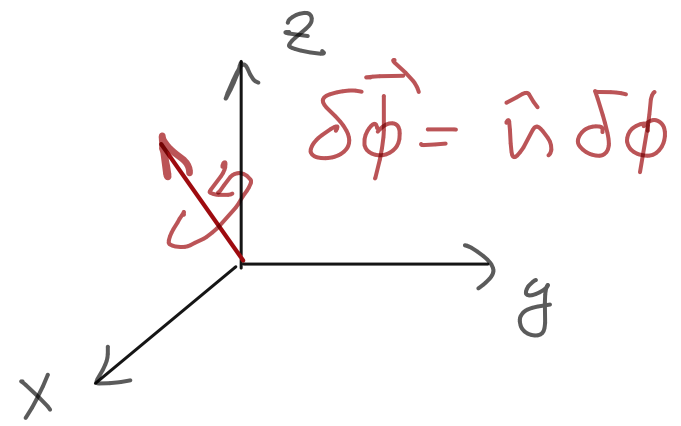
Hamilton function

(363) $$\boxed{H \equiv \sum_j \frac{\partial L}{\partial \dot{\vec{q}}_j} \dot{\vec{q}}_j - L \qquad H = \text{const.}}$$

conserved: Hamilton function

(c) Isotropy in space

Let us consider infinitesimal rotations in space



(364) $$\delta\vec{\phi} = (\delta\phi_x, \delta\phi_y, \delta\phi_y)$$

(365) $$\delta\vec{r}_j = \delta\vec{\phi} \times \vec{r}_j$$

and thus

$$\delta\dot{\vec{r}}_j = \delta\vec{\phi} \times \dot{\vec{r}}_j$$

Let us assume that L is invariant under these infinitesimal rotations

(366) $$\delta L = \sum_j \frac{\partial L}{\partial \vec{r}_j} \cdot \delta\vec{r}_j + \frac{\partial L}{\partial \dot{\vec{r}}_j} \cdot \delta\dot{\vec{r}}_j \overset{!}{=} 0$$

for $\forall\, \delta\vec{\phi}$

canonical momenta

(367) $$\vec{\pi}_j = \frac{\partial L}{\partial \dot{\vec{r}}_j}$$

$$\frac{d}{dt}\vec{\pi}_j = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_j} \underset{\text{Euler Lagrange}}{=} \frac{\partial L}{\partial \vec{r}_j}$$

thus

$$\delta L = \sum_j \left[\dot{\vec{\pi}}_j \cdot (\delta\vec{\phi} \times \delta\vec{r}_j) + \vec{\pi}_j \cdot (\delta\vec{\phi} \times \dot{\vec{r}}_j) \right] = 0$$

$$= \delta\vec{\phi} \cdot \frac{d}{dt} \sum_j (\vec{r}_j \times \vec{\pi}_j) = \delta\vec{\phi} \cdot \dot{\vec{L}}$$

$$\Rightarrow \quad \boxed{\vec{L} = \text{const}} \qquad (368)$$

conserved: total angular momentum

All of the above considered transformations form a continuous group

(D) a reminder: groups

Def: The elements $\{a, b, c, \ldots\}$ of a set form a group G if an operation, which we call multiplication exists with the following properties

(369)

(i) $a, b \in G \Rightarrow ab \in G$

(ii) $\exists\, e \in G$ such that $ae = ea = a$

(iii) $\forall a \ \exists\, a^{-1}$ such that $aa^{-1} = a^{-1}a = e$

(iv) $(ab)c = a(bc)$

Def: A group is called abelian if

$$ab = ba \qquad \forall a, b \in G \qquad (370)$$

Def: A group is called continuous, if all elements can be characterized by one or several continuous parameters

Def: A group is called continously connected if every element can be transformed into every other one by continuous variations of parameter

Def: Two groups are called isomorphic if a bijective mapping exists

$$\underset{\in G_1}{\{a, b, c \dots\}} \longleftrightarrow \underset{\in G_2}{\{a', b', c', \dots\}}$$

such that

$$ab \leftrightarrow a'b'$$

Def: If G_1 is isomorphic to a group G_2 whose elements are matrices, G_2 is called matrix representation

VIII.2 Symmetries in quantum mechanics: general notes

Let us ask what happens to a q.m. wavefunction under coordinate transformations? Consider translation first

active interpretation: • coordinate system unchanged
• wavefunction shifted

A , B

$\vec{r}$ $\qquad$ $\vec{r}\,' = \vec{r} + \vec{a}$

(371) $$\psi_A(\vec{r}) \overset{!}{=} \psi_B(\vec{r}\,') = \psi_B(\vec{r}+\vec{a})$$

translation conserves norm and scalar product
→ unitary trafo

$$\psi_B(\vec{r}) = U(\vec{a})\,\psi_A(\vec{r}) \underset{(371)}{=} \psi_A(\vec{r}-\vec{a})$$

$$= \psi_A(\vec{r}) - \vec{a}\cdot\vec{\nabla}\,\psi_A(\vec{r}) + \cdots \quad \text{(Taylor series)}$$

⇒

(372) $$\boxed{U(\vec{a}) = e^{-\vec{a}\,\vec{\nabla}} = e^{-i\frac{\vec{a}\cdot\vec{p}}{\hbar}}} \qquad U^{+} = U^{-1}$$

One recognizes that the $U(\vec{a})$ form an abelian group upon successive application

(373) $$\boxed{\begin{aligned} U(\vec{a})U(\vec{b}) &= e^{-i\frac{\vec{a}\cdot\vec{p}}{\hbar}}\, e^{-i\frac{\vec{b}\cdot\vec{p}}{\hbar}} = e^{-i\frac{(\vec{a}+\vec{b})\cdot\vec{p}}{\hbar}} \\ &= U(\vec{a}+\vec{b}) = U(\vec{b})U(\vec{a}) \end{aligned}}$$

Furthermore there is the isomorphism

$$\{\vec{a}\} \longleftrightarrow \{U(\vec{a})\} \tag{374}$$

Equation of motion of shifted wave functions

$$i\hbar \frac{\partial}{\partial t} \psi_A(\vec{r},t) = H\psi_A(\vec{r},t)$$

$$i\hbar \frac{\partial}{\partial t} \psi_B(\vec{r},t) = i\hbar \frac{\partial}{\partial t} U(\vec{a})\psi_A(\vec{r},t)$$

$$= U(\vec{a}) H \psi_A(\vec{r},t) = U(\vec{a}) H U^{-1}(\vec{a}) U(\vec{a}) \psi_A$$

$$= U(\vec{a}) H U^{-1}(\vec{a}) \psi_B(\vec{r},t)$$

i.e.

$$\boxed{H_B = U(\vec{a}) H_A U^{-1}(\vec{a})} \tag{375}$$

Thus the equations of motion are invariant under the translation if

$$\boxed{[H, U(\vec{a})] \overset{!}{=} 0} \quad \text{here } [H, \vec{p}] = 0 \tag{376}$$

Def: A group of transformations is called symmetry group if the corresponding unitary representations commute with the Hamiltonian. The group does not have to be continuous

Symmetries and Degeneracies of $\hat{H}$

Let $|\phi_E\rangle$ be an eigenstate of $\hat{H}$ with eigenvalue E
Let $\hat{U}$ be an element of a symmetry group of $\hat{H}$

Then
$$|\phi_E'\rangle = U|\phi_E\rangle$$
is also an eigenstate with eigenvalue E
$$\hat{H}|\phi_E'\rangle = \hat{H}U|\phi_E\rangle = U\hat{H}|\phi_E\rangle = UE|\phi_E\rangle$$
$$= E|\phi_E'\rangle$$

Now, if there are two unitary operators U_1, U_2 with

(377) $$[U_1, U_2] \neq 0 \qquad [U_1, H] = [U_2, H] = 0$$

then there exist (at least) two linear independent eigenstates $|\phi_1\rangle, |\phi_2\rangle$ of $\hat{H}$ with the same eigenvalue $\Longrightarrow$ degeneracy

Symmetry groups

- discrete groups
- continuous groups (Lie groups)
- mixture of both

VIII.3 Lie groups

Def: continuous groups whose elements are unitary operators

$$\hat{U}(a_1, a_2, \ldots)$$

which are analytic in n continuous parameters a_i, $i = 1, 2, \ldots n$, which uniquely define the elements are called Lie groups

we set

(378) $$\boxed{\hat{U}(0) \overset{!}{=} \mathbb{1}}$$

Def: The n operators

(379) $$\boxed{\hat{L}_\mu = i \left. \frac{\partial \hat{U}(a_j)}{\partial a_\mu} \right|_{\{a_j\}=0}}$$

are called generators of the Lie group

Their relevance can be seen if we consider infinitesimal transformations around the identity

(380)
$$\hat{U}(\delta\alpha_\mu) = \hat{U}(0) + \left. \frac{\partial \hat{U}}{\partial \alpha_\mu} \right|_{\{\alpha_j\}=0} \delta\alpha_\mu$$
$$= \mathbb{1} - i \hat{L}_\mu \delta\alpha_\mu$$

Making use of the group character we can construct finite transformations

(381) $$\alpha_\mu = N\,\delta\alpha_\mu$$

by successive infinitesimal transformations

$$\hat{U}(\alpha_\mu) = \left[\hat{U}(\delta\alpha_\mu)\right]^N = \left[1 - i\hat{L}_\mu \delta\alpha_\mu\right]^N$$

$$= \left[1 - i\hat{L}_\mu \frac{\alpha_\mu}{N}\right]^N \xrightarrow[N\to\infty]{} e^{-i\hat{L}_\mu \alpha_\mu}$$

(382) $$\boxed{\hat{U}(\alpha_\mu) = e^{-i\hat{L}_\mu \alpha_\mu}}$$

If α_μ is real, i.e. $\alpha_\mu = \alpha_\mu^*$ then

(383) $$\boxed{\hat{L}_\mu^+ = \hat{L}_\mu}$$ are hermitian

- Let us consider $U^{-1}(\delta\alpha_\mu)$ and $U(\delta\beta_\mu)$ in second order

$$U^{-1}(\delta\alpha_\mu) = \mathbb{1} + i\sum_\mu \hat{L}_\mu \delta\alpha_\mu - \frac{1}{2}\sum_{\mu\nu} \hat{L}_\mu \hat{L}_\nu \delta\alpha_\mu \delta\alpha_\nu$$

$$U(\delta\beta_\mu) = \mathbb{1} - i\sum_\mu \hat{L}_\mu \delta\beta_\mu - \frac{1}{2}\sum_{\mu\nu} \hat{L}_\mu \hat{L}_\nu \delta\beta_\mu \delta\beta_\nu$$

then (drop "sum")

$$\mathcal{U}^{-1}(\delta\beta_\mu)\,\mathcal{U}^{-1}(\delta\alpha_\mu)\,\mathcal{U}(\delta\beta_\mu)\,\mathcal{U}(\delta\alpha_\mu) =$$

$$= \left(\mathbb{1} + i\hat{L}_\mu \delta\beta_\mu - \frac{1}{2}\hat{L}_\nu \hat{L}_\lambda \delta\beta_\nu \delta\beta_\lambda\right)$$

$$\cdot \left(1 + i\hat{L}_\sigma \delta\alpha_\sigma - \frac{1}{2}\hat{L}_\sigma \hat{L}_\gamma \delta\alpha_\sigma \delta\alpha_\gamma\right)$$

$$\cdot \left(\mathbb{1} - i\hat{L}_\varepsilon \delta\beta_\varepsilon - \frac{1}{2}\hat{L}_\omega \hat{L}_\varepsilon \delta\beta_\omega \delta\beta_\varepsilon\right)$$

(384)
$$\cdot \left(\mathbb{1} - i\hat{L}_\eta \delta\alpha_\eta - \frac{1}{2}\hat{L}_\eta \hat{L}_\tau \delta\alpha_\eta \delta\alpha_\tau\right)$$

$$= \dots =$$

$$= \mathbb{1} + \delta\alpha_\mu \delta\beta_\nu \left(\hat{L}_\mu \hat{L}_\nu - \hat{L}_\nu \hat{L}_\mu\right)$$

Since the succession of infinitesimal unitary trafo's must be again a infinitesimal unitary trafo

(385)
$$\delta\alpha_\mu \delta\beta_\nu \left(\hat{L}_\mu \hat{L}_\nu - \hat{L}_\nu \hat{L}_\mu\right) \stackrel{!}{=} -i\delta\gamma_\lambda \hat{L}_\lambda$$

with

(386)
$$-i\delta\gamma_\lambda = C_{\mu\nu\lambda}\, \delta\alpha_\mu \delta\beta_\nu$$

we find

(387) $$\boxed{[\hat{L}_\mu, \hat{L}_\nu] = C_{\mu\nu\lambda} \hat{L}_\lambda}$$

Lie algebra

The coefficients $C_{\mu\nu\lambda}$ are called structure factors

The Lie algebra is closed under commutation.

Jacobi-identity

(388) $$[[\hat{L}_i, \hat{L}_j], \hat{L}_k] + [[\hat{L}_j, \hat{L}_k], \hat{L}_i] + [[\hat{L}_k, \hat{L}_i], \hat{L}_j] = 0$$

This implies

(389) $$\boxed{C_{ijm} C_{mkn} + C_{jkm} C_{min} + C_{kim} C_{mjn} = 0}$$

Def: The number of commuting generators of a Lie group is called the rank of the group

example:

- translations in $\mathbb{R}^3$ rank 3
  $[p_x, p_y] = [p_x, p_z] = [p_y, p_z] = 0$
- rotations in $\mathbb{R}^3$ rank 1