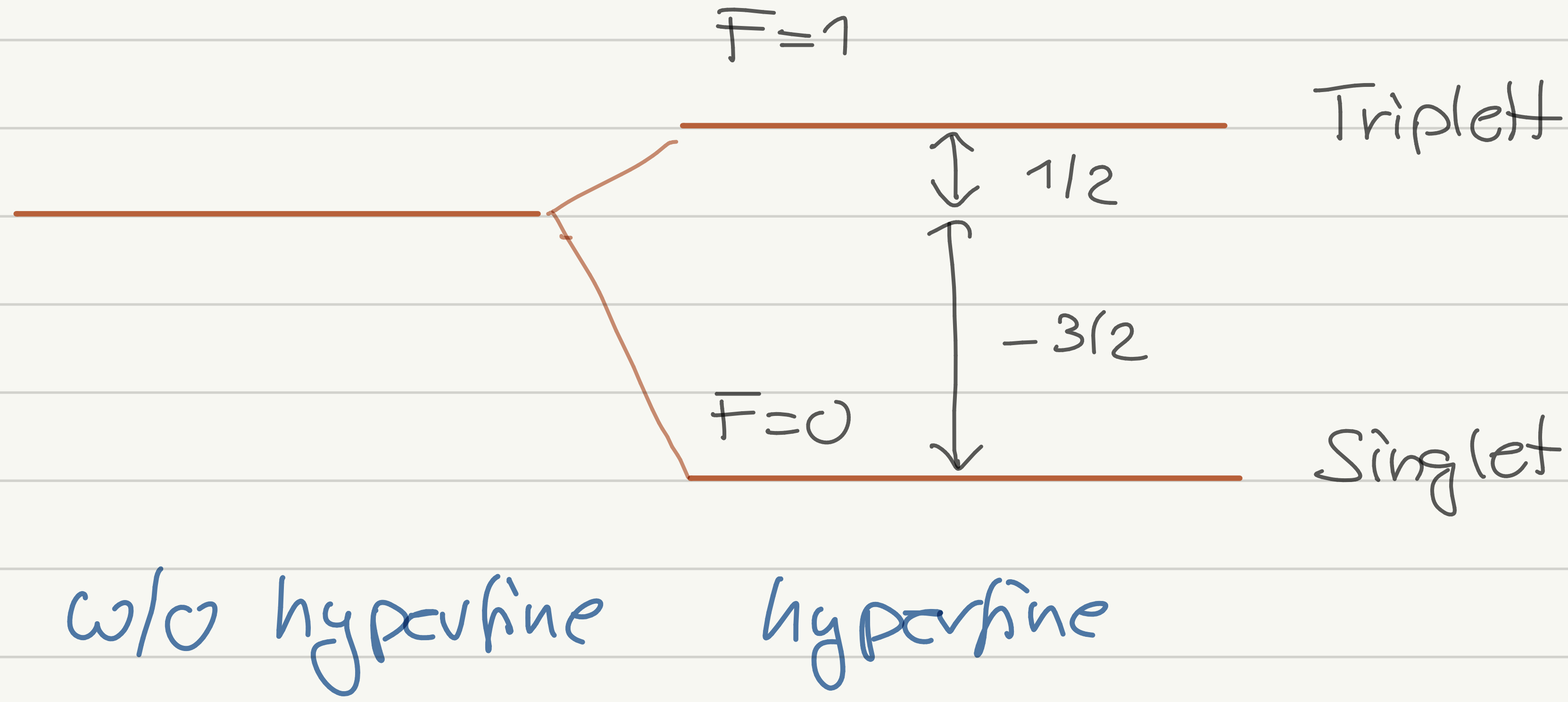
eigenvalues of

$$2\vec{S}\cdot\vec{I} = \begin{cases} 1(1+1) - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{1}{2} \\ 0 - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right) = -\frac{3}{2} \end{cases}$$

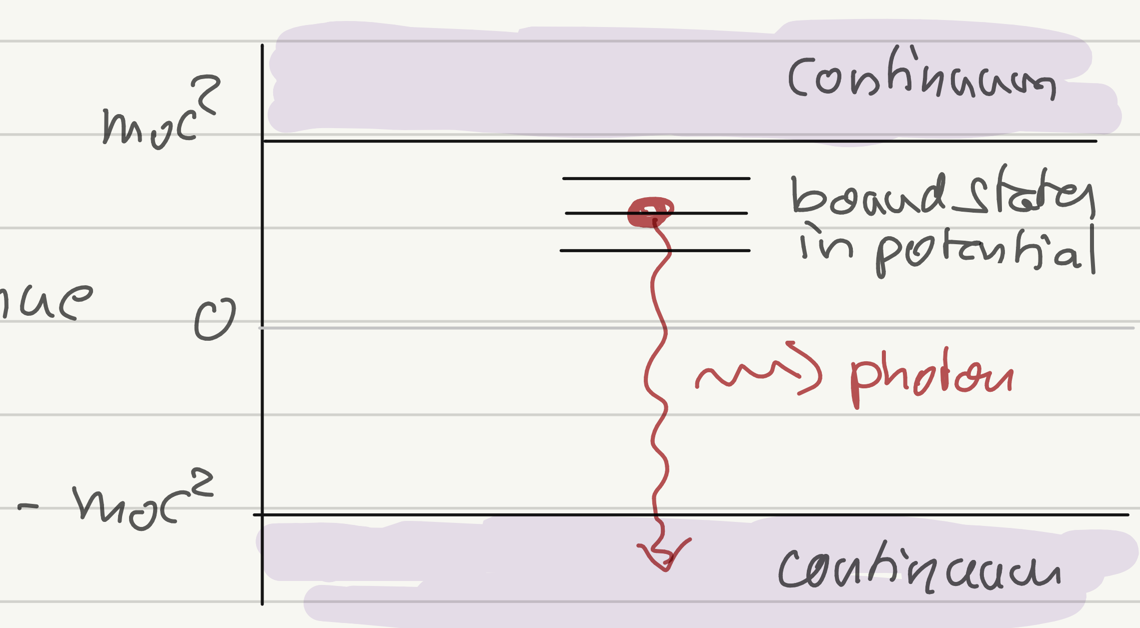
(332)




# VII.6 Dirac sea

The unboundedness from below of the energy spectrum of the Dirac equation creates a problem:

an electron can emitt elm. energy an continue to fall down in the spectrum

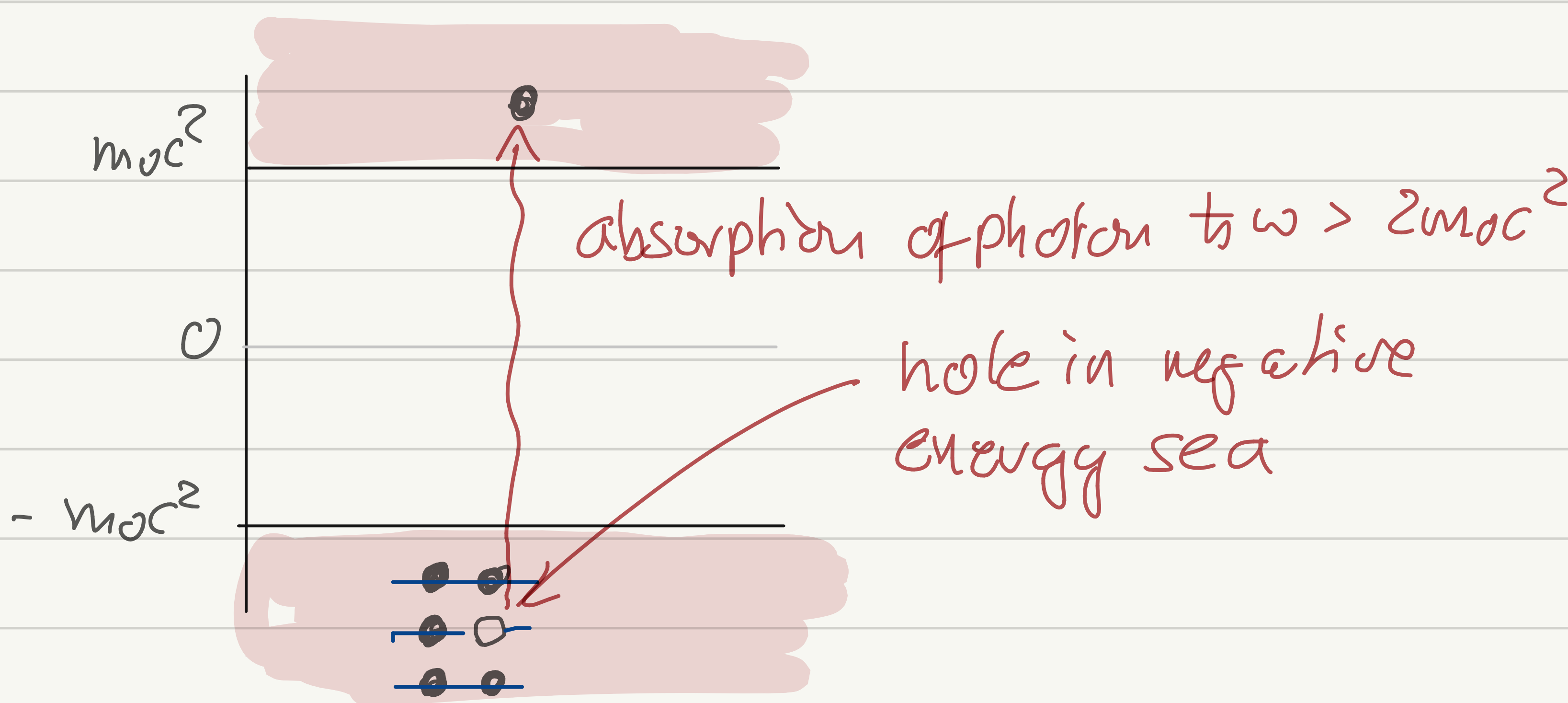


## Solution: Dirac sea

"vacuum" state

$$\boxed{\begin{array}{l}\text{all states with negative energy are occupied}\\ \text{and the particles are Fermions i.e. obey the}\\ \text{Pauli exclusion principle}\end{array}}$$

consequences




generation of a pair of particle and "hole"

energy of hole

(333) $$E_{hole_p} = E_{vacuum} - (-E_p) = E_p$$

if we set $E_{vacuum} = 0$

(334) $$\boxed{E_{positron} = +c\sqrt{\vec{p}^{\,2} + m_0^2c^2}}$$

hole $\hat{=}$ anti-particle with positive En.

## momentum of hole

(335) $$\boxed{\vec{P}_{positron} = \vec{P}_{vacuum} - \vec{P}_{el} = -\vec{P}_{el}}$$

so $\hbar\vec{k} = \vec{P}_{pos} + \vec{P}_{el}$ (pair creation)

## charge of hole

$$q_{positron} = Q_{vac} - q_{el}$$

(336) $$\boxed{q_{positron} = -q_{electron}}$$

## velocity

$$\left[\frac{d\vec{x}}{dt}\right] = c[\vec{\alpha}] = \frac{c^2\vec{p}\,\Lambda}{c\sqrt{\vec{p}^2 + m_0^2c^2}}$$

for negative energy solutions

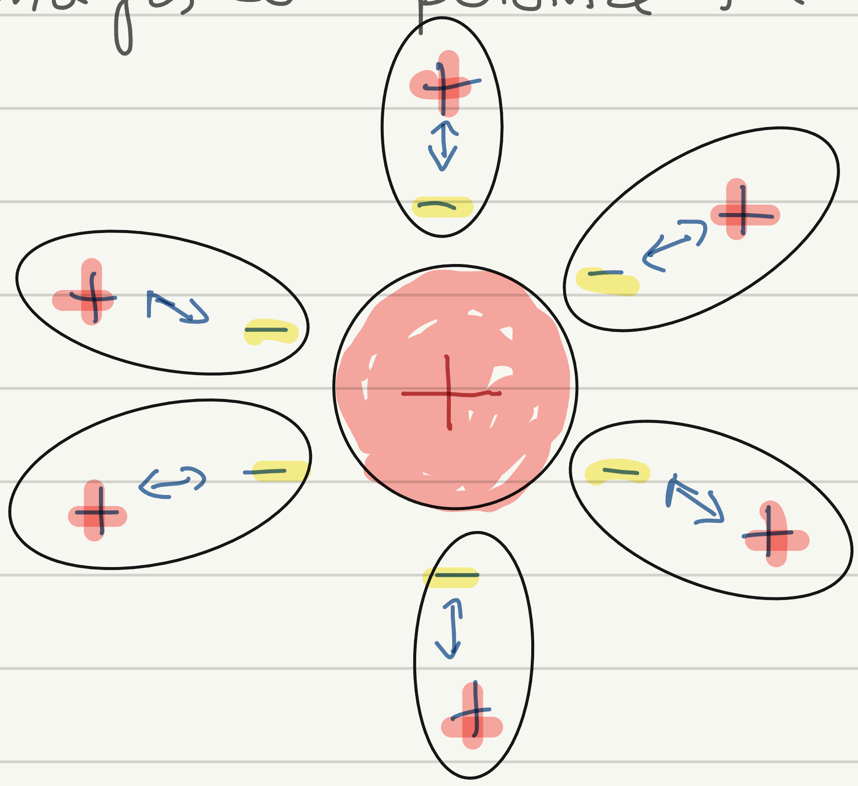
$$\left[\frac{d\vec{x}}{dt}\right] \sim -c\vec{p}$$

for a positron (hole)

(337) $$\boxed{\left[\frac{d\vec{x}}{dt}\right] \sim c\vec{p}_{positron}}$$

# vacuum-polarization

electr. charges can polarize the vacuum



| |
|---|
| hole theory is a many-particle theory and we have to give up the interpretation of the Dirac wavefunction as single-particle probability amplitude |

## (A) Charge conjugation

particle charge $q$ $\underset{?}{\overset{\text{new symmetry}}{\longleftrightarrow}}$ antiparticle charge $-q$

(338)
$$\left(i\hbar \not{\partial} - \frac{e}{c}\not{A} - m_0 c\right)\psi = 0 \qquad \text{electron}$$
$$\left(i\hbar \not{\partial} + \frac{e}{c}\not{A} - m_0 c\right)\psi_c = 0 \qquad \text{positron}$$

(339)
$$\psi_c \overset{?}{\longleftrightarrow} \psi$$

now

$$\left(i\hbar\frac{\partial}{\partial x^\mu}\right)^* = -i\hbar\frac{\partial}{\partial x^\mu} \qquad \text{but} \quad A_\mu^* = A_\mu$$

i.e.

$$\left[\left(i\hbar\cancel{\partial} - \frac{e}{c}\cancel{A} - m_0c\right)\psi\right]^* =$$

$$= \left[\left(i\hbar\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right)\gamma^\mu\psi - m_0c\psi\right]^* = 0$$

$$\Rightarrow \left[\left(i\hbar\frac{\partial}{\partial x^\mu} + \frac{e}{c}A_\mu\right)\gamma^{\mu*} + m_0c\right]\psi^* = 0$$

$\Rightarrow$ consider

(340) $$U = C\gamma^0$$

then we require

(341) $$U\gamma^{\mu*}U^{-1} \overset{!}{=} -\gamma^\mu$$

since then

$$\left[\left(i\hbar\frac{\partial}{\partial x^\mu} + \frac{e}{c}A_\mu\right)U\gamma^{\mu*}U^{-1} + m_0c\right]U\psi^* = 0$$

i.e. $$\left[\left(i\hbar\frac{\partial}{\partial x^\mu} + \frac{e}{c}A_\mu\right)\gamma^\mu - m_0c\right]U\psi^* = 0$$

Dirac eq.!

thus

(342) $$\boxed{\psi_c = U\psi^* = C\gamma^0\psi^* = C\bar{\psi}^T}$$

charge conjugation

Dirac equation with "+e" for $\psi$

$\Longleftrightarrow$ Dirac equation with "−e" for $\psi_c$

What is C ?

To fulfill (341)

$$-\gamma^\mu = U\gamma^{\mu *}U^{-1} = C\gamma^0\gamma^{\mu *}(C\gamma^0)^{-1}$$

$$= C\gamma^0\gamma^{\mu *}\gamma^0 C^{-1}$$

now we remember

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^0 = \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

i.e.

(343) $$\gamma^0\gamma^{\mu *}\gamma^0 = \gamma^{\mu T}$$

i.e.

$$-\gamma^{\mu T} = (C^{-1})^T\gamma^\mu C^T$$

since $\gamma^{1T} = -\gamma^1 \qquad \gamma^{2T} = \gamma^2 \qquad \gamma^{3T} = -\gamma^3$

$\gamma^{0T} = \gamma^0$

follows

(344) $$[C^T, \gamma^1] = [C^T, \gamma^3] = 0$$
$$\{C^T, \gamma^0\} = \{C^T, \gamma^2\} = 0$$

this then gives for the 4x4 matrix $C$

$$C = i\gamma^2\gamma^0 = -C^{-1} = -C^+ = -C^T$$
$$\psi_c = U\psi^* = C\gamma^0\psi^* = C\gamma^0 K\psi$$

so

(345) $$\boxed{\psi_c = i\gamma^2 K\psi}$$

charge conjugation
of Dirac bispinor

it holds

(346) $$(\psi_c)_c = (i\gamma^2\psi^*)_c = i\gamma^2(i\gamma^2\psi^*)^*$$
$$= \gamma^2\gamma^{2*}\psi = \psi \quad \checkmark$$

expectation values in charge conjugated state

$$\langle\hat{O}\rangle_c = \int d^3r\, \psi_c^+ \hat{O}\, \psi_c$$

$$= \int d^3r\, (i\gamma^2 \psi^*)^\dagger\, \hat{O}\, (i\gamma^2 \psi^*)$$

$$= \int d^3r\ \psi^{*\dagger} \gamma^{2\dagger}\, \hat{O}\, \gamma^2\, \psi^*$$

(343) $$= \int d^3r\ \psi^{*\dagger} \gamma^0 \gamma^2 \gamma^0\, \hat{O}\, \gamma^2\, \psi^*$$

$$= -\int d^3r\ \psi^{*\dagger} \gamma^2\, \hat{O}\, \gamma^2 \psi^*$$

$$= -\left[\int d^3r\ \psi^\dagger (\gamma^2 \hat{O} \gamma^2)^* \psi\right]^*$$

$$= -\langle \gamma^2 \hat{O} \gamma^2 \rangle^*$$

i.e.

(347) $$\boxed{\langle \hat{O} \rangle_c = -\langle \gamma^2 \hat{O} \gamma^2 \rangle^*}$$

examples:

$$\langle \beta \rangle_c = -\langle \beta \rangle \qquad \langle \alpha_i \rangle_c = \langle \alpha_i \rangle$$

(348) $$\langle \hat{\vec{x}} \rangle_c = \langle \hat{\vec{x}} \rangle \qquad \langle \hat{\vec{p}} \rangle_c = -\langle \hat{\vec{p}} \rangle$$

$$\langle \hat{\vec{\Sigma}} \rangle_c = -\langle \hat{\vec{\Sigma}} \rangle \qquad \langle \hat{\vec{L}} \rangle_c = -\langle \hat{\vec{L}} \rangle$$

$$\langle \hat{\vec{J}} \rangle_c = -\langle \hat{\vec{J}} \rangle$$

# (B) Time reversal

(349) $$\boxed{\psi'(\vec{r},t') \equiv \hat{T}\psi(\vec{r},t) \equiv \psi'(\vec{r},-t)}$$

T must be antiunitary

(350) $$\boxed{TiT^{-1} = -i}$$

then for a time reversal (TR) invariant $\hat{H}$

(351) $$THT^{-1} = H$$

(352) $$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

$$Ti\hbar\frac{\partial}{\partial t}T^{-1}|\psi'\rangle = THT^{-1}|\psi'\rangle$$

$$-i\hbar\frac{\partial}{\partial(-t)}|\psi'\rangle = H|\psi'\rangle$$ identical to (352)

(353) $$\boxed{T = UK}$$ $$U^{+} = U^{-1}$$

homework $$U\vec{\alpha}U^{-1} \stackrel{!}{=} -\vec{\alpha} \qquad U\beta U^{-1} = \beta$$

with this

(354) $$\boxed{T = -i\alpha_1\alpha_3 K}$$

time inversion

## (c) Parity inversion and PCT

we had already discussed the parity-trafo operator

(355) $$\boxed{\hat{P}\psi = e^{i\varphi}\gamma^0\psi}$$

which leaves Dirac equation invariant

One finds that the combination of parity trafo, time reversal and charge conjugation is again a symmetry of the Dirac theory

(356) $$\psi_{PCT}(x) = PCT\,\psi(x) = i e^{i\varphi}\gamma_5\,\psi(-x)$$

is again solution of DE !

This is a general feature of relativistic theories

PCT symmetry