bilinear forms

(530)

(i)	scalar	$\bar{\psi}\psi$
(ii)	pseudoscalar	$\bar{\psi}\gamma_5\psi$
(iii)	vector	$\bar{\psi}\gamma^\mu\psi$
(iv)	pseudovector	$\bar{\psi}\gamma_5\gamma^\mu\psi$
(v)	tensor rank 2	$\bar{\psi}\sigma^{\mu\nu}\psi$

VII.4 Solution of Dirac equation for some simple problems

(A) Plane waves generated by Lorentz trafo

For a Dirac particle at rest we have

(531) $$\psi^r(\vec{r},t) = \omega^r(0)\, e^{-i\varepsilon_r m_0 c^2 t/\hbar}$$

$$r = 1,2,3,4$$

$$\varepsilon_{1,2} = +1 \qquad \varepsilon_{3,4} = -1$$

$$\omega^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \omega^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \omega^3(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \omega^4(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now we can make use of a Lorentz trafo to generate the general solution of a free Dirac particle at any momentum

L Trafo to frame with $\vec{v}$

$$S(-\vec{v}) = e^{-\frac{v}{2c}\vec{\alpha}\cdot\frac{\vec{v}}{v}}$$

(532)

$$= \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{bmatrix} 1 & 0 & \frac{p_z c}{E+m_0c^2} & \frac{p_- c}{E+m_0c^2} \\ 0 & 1 & \frac{p_+ c}{E+m_0c^2} & \frac{-p_z c}{E+m_0c^2} \\ \frac{p_z c}{E+m_0c^2} & \frac{p_- c}{E+m_0c^2} & 1 & 0 \\ \frac{p_+ c}{E+m_0c^2} & \frac{-p_z c}{E+m_0c^2} & 0 & 1 \end{bmatrix}$$

where

(533)
$$p_\pm \equiv p_x \pm i p_y$$

Here we made use of

$$e^{-\frac{v}{2c}\vec{\alpha}\cdot\frac{\vec{v}}{c}} = 1 - \frac{v}{2c}\frac{\vec{\alpha}\cdot\vec{v}}{v} + \frac{1}{2!}\frac{v^2}{4c^2}\frac{(\vec{\alpha}\cdot\vec{v})^2}{v^2} - + \dots$$

where

$$(\vec{\alpha}\cdot\vec{v})^2 = \alpha^i\alpha^k v_i v_k = \underbrace{\gamma^0\gamma^i\gamma^0}_{-\gamma^i}\gamma^k v_i v_k$$

$$= -\gamma^i\gamma^k v_i v_k = -\frac{1}{2}\left(\gamma^i\gamma^k v_i v_k + \gamma^j\gamma^k v_k v_i\right)$$

anti-commutator

$$= -\frac{1}{2}\left(2g^{ij} v_i v_j\right) = v^2$$

I.e. the series separates into even and odd powers

$$(534)\quad S = \mathbb{1}\cosh\frac{v}{2c} + \frac{\vec{\alpha}\cdot\vec{v}}{v}\sinh\frac{v}{2c}$$

using

$$\frac{\vec{\alpha}\cdot\vec{v}}{c} = \alpha_x\frac{v_x}{v} + \alpha_y\frac{v_y}{v} + \alpha_z\frac{v_z}{v}$$

$$= \frac{p_x}{p}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \frac{i p_y}{p}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \frac{p_z}{p}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{p}\begin{bmatrix} 0 & 0 & p_z & p_- \\ 0 & 0 & p_+ & -p_z \\ p_z & p_- & 0 & 0 \\ p_+ & -p_z & 0 & 0 \end{bmatrix}$$

$$(535)\quad \boxed{\begin{aligned} \omega^r(\vec{p}) &= S\left(-\frac{\vec{p}}{E}\right)\omega^r(0) \\ \underline{\psi}^r(\vec{r},t) &= \omega^r(\vec{p})\, e^{-i\varepsilon_r \frac{p_\mu x^\mu}{\hbar}} \end{aligned}}$$

general solution of free DE

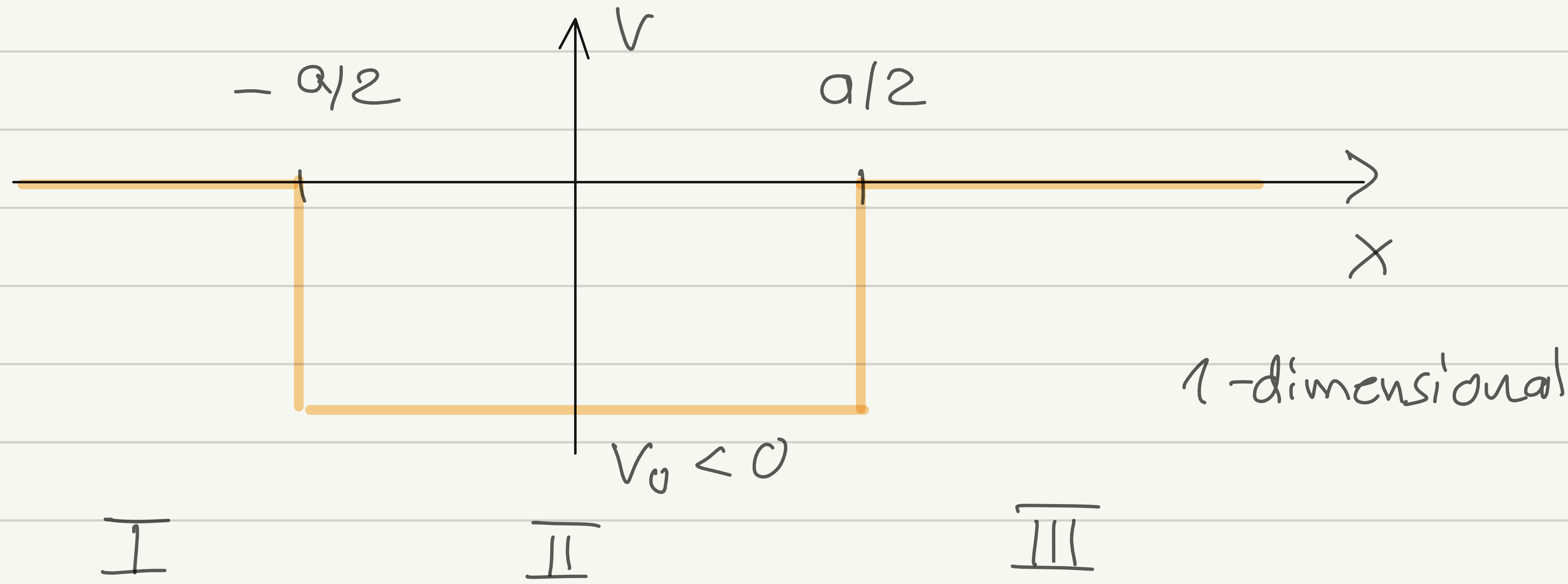
The coefficients $\omega^r(\vec{p})$ fulfill the "algebraic" DE

$$(536)\quad \boxed{\begin{aligned} (\not{p} - \varepsilon_r m_0 c)\,\omega^r(\vec{p}) &= 0 \\ \overline{\omega^r(\vec{p})}\,(\not{p} - \varepsilon_r m_0 c) &= 0 \end{aligned}}$$

(Here $\not{p}$ not $\hat{\not{p}}$!)

The general solution can we written as superposition of the ψ^r

(B) Dirac particle in a 1-dimensional finite potential well



stationary solution $\quad \psi = \phi e^{-iEt/\hbar}$

$$(537)\quad \Big[c\vec{\alpha}\cdot(\vec{p} - \tfrac{e}{c}\underbrace{\vec{A}}_{\equiv 0}) + \underbrace{eA_0}_{=V} + \beta m_0 c^2\Big]\varphi = E\varphi$$

I, III

(538) $$\left(c\alpha_x p_x + \beta m_0 c^2\right)\varphi = E\varphi$$

assume spin polarization "+" see eqs. (445)

solution:

(539) $$\varphi_{I,III} = A_{I,III}\, e^{ik_1 x}\begin{pmatrix}1\\0\\\frac{\hbar k_1 c}{E+m_0c^2}\\0\end{pmatrix} + B_{I,III}\, e^{-ik_1 x}\begin{pmatrix}1\\0\\\frac{-k_1 c\hbar}{E+m_0c^2}\\0\end{pmatrix}$$

$$\hbar^2 k_1^2 = \frac{E^2}{c^2} - m_0^2c^2$$

II

$$\left(c\alpha_x p_x + \beta m_0 c^2\right)\varphi = \left(E - V_0\right)\varphi$$

solution

(540) $$\varphi_{II} = A_{II}\, e^{ik_2 x}\begin{pmatrix}1\\0\\\frac{\hbar k_2 c}{E-V_0+m_0c^2}\\0\end{pmatrix} + B_{II}\, e^{-ik_2 x}\begin{pmatrix}1\\0\\\frac{-\hbar k_2 c}{E-V_0+m_0c^2}\\0\end{pmatrix}$$

$$\hbar^2 k_2^2 = \frac{(E-V_0)^2}{c^2} - m_0^2 c^2$$

boundary conditions

$\partial_\mu j^\mu = 0$ wavefunction must be smooth

(5.41)
$$\left.\begin{aligned} \phi_I\left(-\tfrac{a}{2}\right) &\overset{!}{=} \phi_{II}\left(-\tfrac{a}{2}\right) \\ \phi_{II}\left(\tfrac{a}{2}\right) &\overset{!}{=} \phi_{III}\left(\tfrac{a}{2}\right) \end{aligned}\right\}$$
2 complex equations $\Rightarrow$ 4 conditions

(5.42)
$$\int d^3r \, \Psi^+(r) \Psi(r) = 1$$
1 condition

assumption of bound states

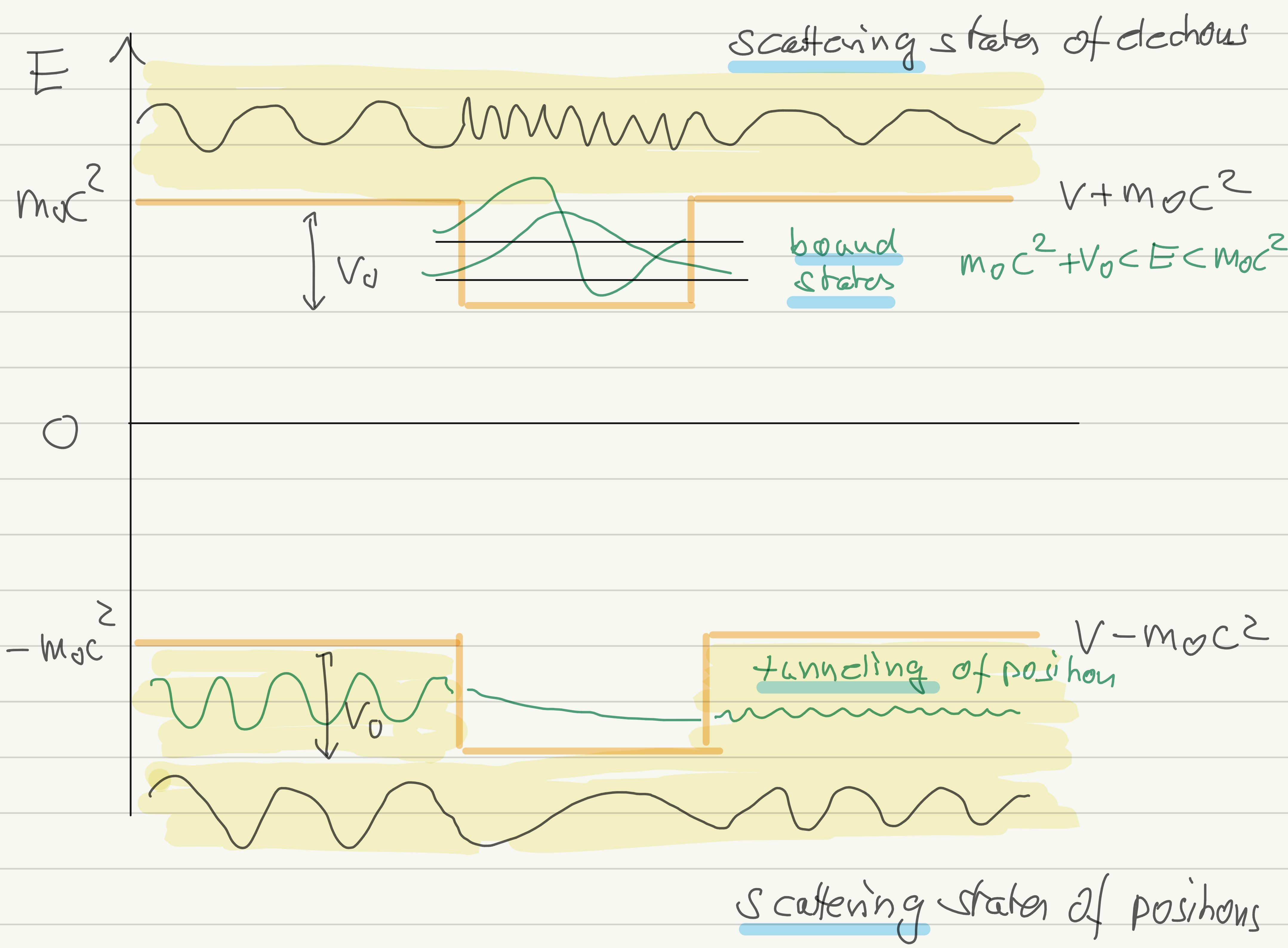
(5.43)
$$\phi_I(x \to -\infty) \to 0 \qquad \phi_{III}(x \to +\infty) \to 0$$

2 conditions

$\rightarrow$ only discrete energies E

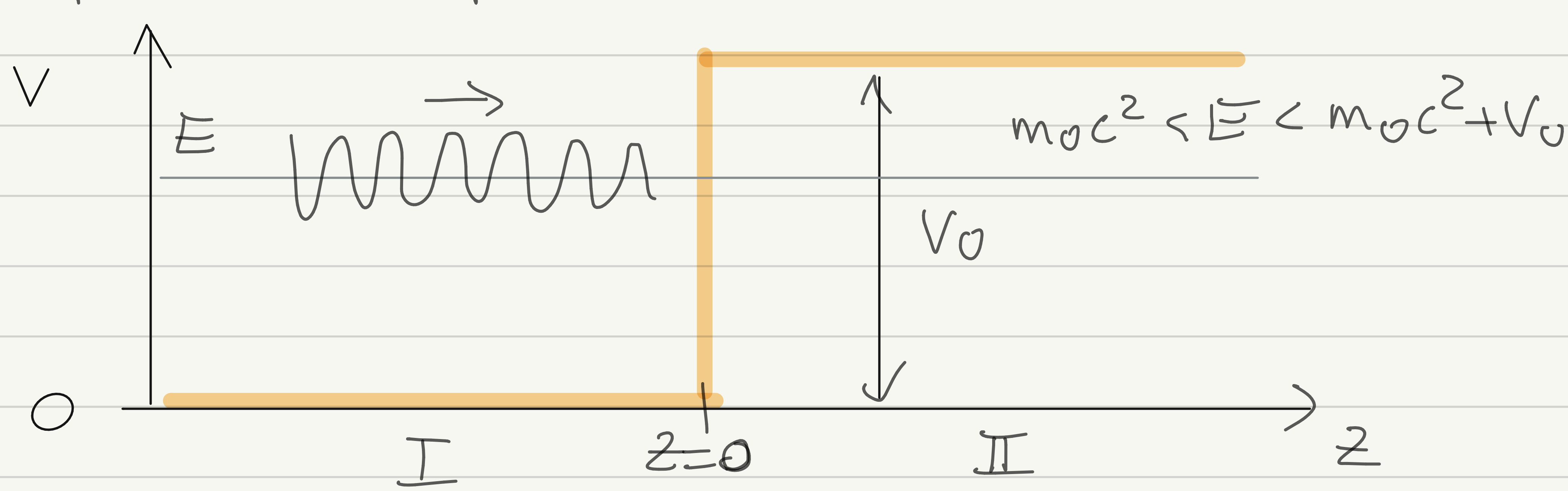
We will not give the explicit solution here which is however straightforward but requires solution of transcendential algebraic equation.

qualitative discussion of solutions $|V_0| < 2m_0c^2$



(C) Klein paradox

consider a Dirac particle impinging on a potential step



stationary Dirac equation

$\boxed{\mathrm{I}}$

$$\psi_I = \psi^i + \psi^r$$

(544) $$= \phi_{in}\, e^{\frac{i}{\hbar}(pz - Et)} + \phi_r\, e^{\frac{i}{\hbar}(-pz - Et)}$$

in stationary DE

$$\left(\frac{E}{c} - \beta m_0 c\right)\psi_I + i\hbar\alpha_z \frac{\partial \psi_I}{\partial z} = 0$$

$\Rightarrow$

(545) $$\left(\frac{E}{c} - \alpha_z p - \beta m_0 c\right)\phi_{in} = 0$$

(546) $$\left(\frac{E}{c} + \alpha_z p - \beta m_0 c\right)\phi_r = 0$$

where

(547) $$\frac{E^2}{c^2} = p^2 + m_0^2 c^2$$

$\boxed{\mathrm{II}}$

$$\psi_{II} = \psi_t = \phi_t\, e^{\frac{i}{\hbar}(\tilde{p}z - Et)}$$

$$\left(\frac{E - V_0}{c} - \beta m_0 c\right)\psi_{II} + i\hbar\alpha_z \frac{\partial \psi_{II}}{\partial z} = 0$$

(548) $$\left(\frac{E - V_0}{c} - \alpha_z \tilde{p} - \beta m_0 c\right)\phi_t = 0$$

where

(549) $$\frac{(E - V_0)^2}{c^2} = \tilde{p}^2 + m_0^2 c^2$$

In order to have a real solution for $\tilde{p}$

(550) $$V_0 \leq E - m_0 c^2 \quad \text{or} \quad V_0 \geq E + m_0 c^2$$

boundary condition at $z=0$

(551) $$\phi_{in} + \phi_r = \phi_t$$

From (545) and (546)

(552) $$\left(\frac{E}{c} - \beta m_0 c\right)(\phi_{in} + \phi_r) = \alpha_z p(\phi_{in} - \phi_r)$$

From (548) and with boundary condition (551):

$$\left(\frac{E}{c} - \beta m_0 c\right)\phi_t = \left(\frac{V_0}{c} + \alpha_z \tilde{p}\right)\phi_t$$

$$= \left(\frac{V_0}{c} + \alpha_z \tilde{p}\right)(\phi_{in} + \phi_r)$$

$$\Rightarrow \quad \left(\frac{V_0}{c} + \alpha_z \tilde{p}\right)(\phi_{in} + \phi_r) = \alpha_z p(\phi_{in} - \phi_r)$$

From this we can extract the relation between ϕ_r and ϕ_{in}

$$\left[\frac{V_0}{c} + \alpha_z(p + \tilde{p})\right]\phi_r = -\left[\frac{V_0}{c} - \alpha_z(p - \tilde{p})\right]\phi_{in}$$

multiply with $\left[\frac{V_0}{c} - \alpha_z(p + \tilde{p})\right]$ from the left yields

$$\phi_r = \frac{-\left[\frac{V_0^2}{c^2} - 2\frac{V_0}{c}\alpha_z p + p^2 - \tilde{p}^2\right]}{\frac{V_0^2}{c^2} - (p+\tilde{p})^2}\,\phi_{in}$$

with (547) and (549)

(553) $$p^2 - \tilde{p}^2 = -\left[\frac{(E-V_0)^2}{c^2} - \frac{E^2}{c^2}\right] = -\frac{V_0^2}{c^2} + \frac{2EV_0}{c}$$

$\Rightarrow$

(554) $$\phi_r = \underbrace{\frac{\frac{2V_0}{c}\left[-\frac{E}{c} + \alpha_z p\right]}{V_0^2/c^2 - (p+\tilde{p})^2}}_{\text{reflection amplitude}}\,\phi_{in} = r\,\phi_{in}$$

- To calculate the reflected intensity we need $\phi_r^+\phi_r$

(555) $$\phi_r^+\phi_r = \left[\frac{\frac{2V_0}{c}}{\frac{V_0^2}{c^2} - (p+\tilde{p})^2}\right]^2 \phi_{in}^+\left(-\frac{E}{c} + \alpha_z p\right)^2\phi_{in}$$

(c number) (matrix)

now $$\left(-\frac{E}{c} + \alpha_z p\right)^2 = \frac{E^2}{c^2} + p^2 - \frac{2E}{c}\alpha_z p$$

so we need $$\phi_{in}^+\alpha_z\phi_{in}$$

From the stationary DE for ϕ_e, eq. (545)

$$\phi_{in}^{+}\alpha_z \cdot \Big| \quad \left(\frac{E}{c} - \alpha_z p - \beta m_0 c\right)\phi_{in} = 0$$

$$\Rightarrow \quad \frac{E}{c}\phi_{in}^{+}\alpha_z\phi_{in} - \phi_{in}^{+}\phi_{in}\,p - \phi_{in}^{+}\alpha_z\beta m_0 c\,\phi_{in} = 0$$

$$\cdot\, \alpha_z\phi_{in} \Big| \quad \phi_{in}^{+}\left(\frac{E}{c} - \alpha_z p - \beta m_0 c\right) = 0$$

$$\Rightarrow$$

$$\frac{E}{c}\phi_{in}^{+}\alpha_z\phi_{in} - \phi_{in}^{+}\phi_{in}\,p - \phi_{in}^{+}\beta\alpha_z m_0 c\,\phi_{in} = 0$$

remember that $\{\beta, \alpha_z\} = 0$, so

(556) $$\phi_{in}^{+}\alpha_z\phi_{in} = \frac{pc}{E}\phi_{in}^{+}\phi_{in}$$

This gives

$$\phi_r^{+}\phi_r = \left[\frac{2V_0/c}{V_0^2/c^2 - (p+\tilde{p})^2}\right]^2 \underbrace{\left(\frac{E^2}{c^2} + p^2 - 2p^2\right)}_{\frac{E^2}{c^2} - p^2 = m_0^2 c^2} \phi_{in}^{+}\phi_{in}$$

(557) $$\phi_r^{+}\phi_r = \left[\frac{2V_0 m_0}{V_0^2/c^2 - (p+\tilde{p})^2}\right]\phi_{in}^{+}\phi_{in}$$

reflection coefficient

(558) $$R = \left[\frac{2V_0 m_0}{V_0^2/c^2 - (p+\tilde{p})^2} \right]^2$$

- so if $V_0 = 0$ $\quad \underline{\underline{R=0}}$ $\quad$ no reflection ✓
- if $V_0 = E - m_0c^2$ $\quad \frac{(E-V_0)^2}{c^2} - m_0^2c^2 = \tilde{p}^2$

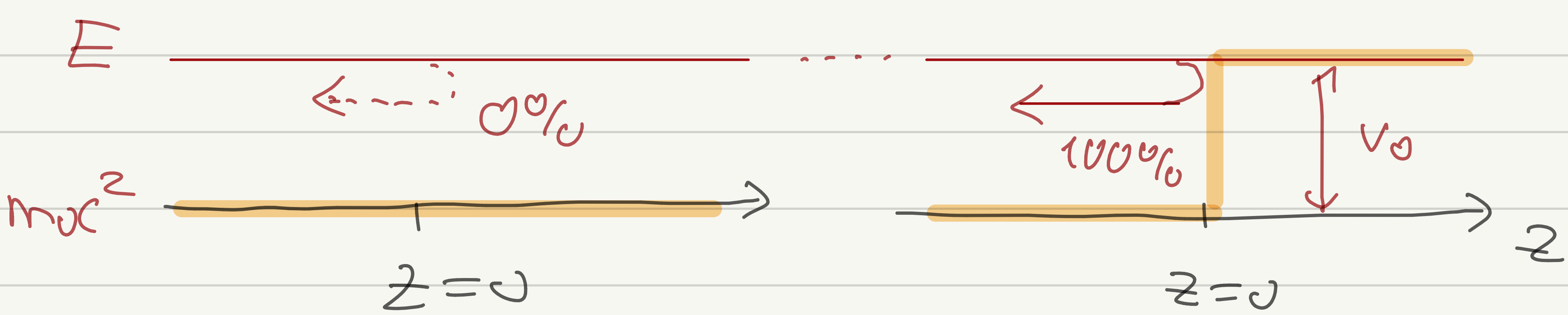
$$\Rightarrow \tilde{p} = 0 \qquad \frac{E^2}{c^2} - m_0^2c^2 = p^2$$

$$p^2 = \left(\frac{E}{c} - m_0c\right)\left(\frac{E}{c} + m_0c\right) = \frac{V_0}{c}\left(\frac{E}{c} + m_0c\right)$$

$$\frac{V_0^2}{c^2} = \frac{V_0}{c}\left[\frac{E}{c} - m_0c\right]$$

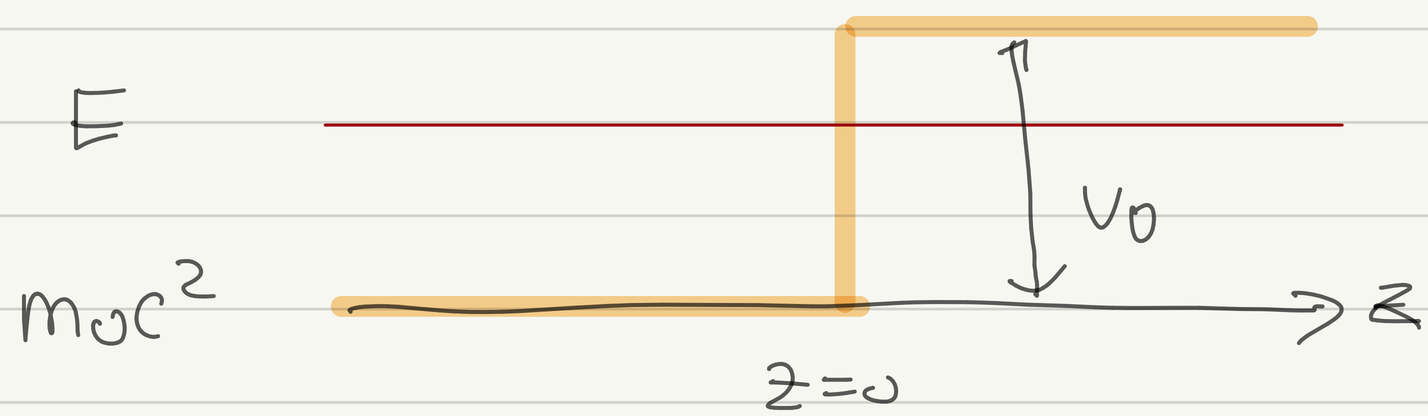
$$\frac{V_0^2}{c^2} - p^2 = \frac{V_0}{c}\left(\frac{E}{c} - m_0c - \frac{E}{c} - m_0c\right) = -2V_0m_0$$

$$\underline{\underline{R=1}}$$



$V_0 = 0$ $\qquad\qquad$ $V_0 = E - m_0c^2$

What happens if $V_0 > E - m_0c^2$



(559) $$\tilde{p}^2 = p^2 - \frac{V_0(2E - V_0)}{c^2}$$

$\Rightarrow$

$\tilde{p}$ is imaginary i.e. $\tilde{p} = i\hbar\mu$ for

(i) $E + m_0c^2 > V_0 > E - m_0c^2$

there

(560) $$\psi_{II} = \phi_+ e^{-\mu z - i\frac{E}{\hbar}t}$$

exponentially decaying amplitude as expected from non-relativistic QM

$\Rightarrow$ $$R = 1$$

however

$\tilde{p}$ becomes real again for

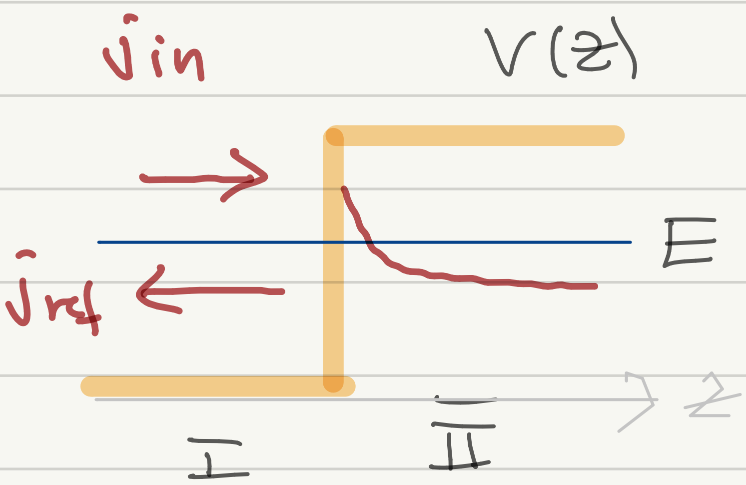
(ii) $V_0 > E + m_0c^2$

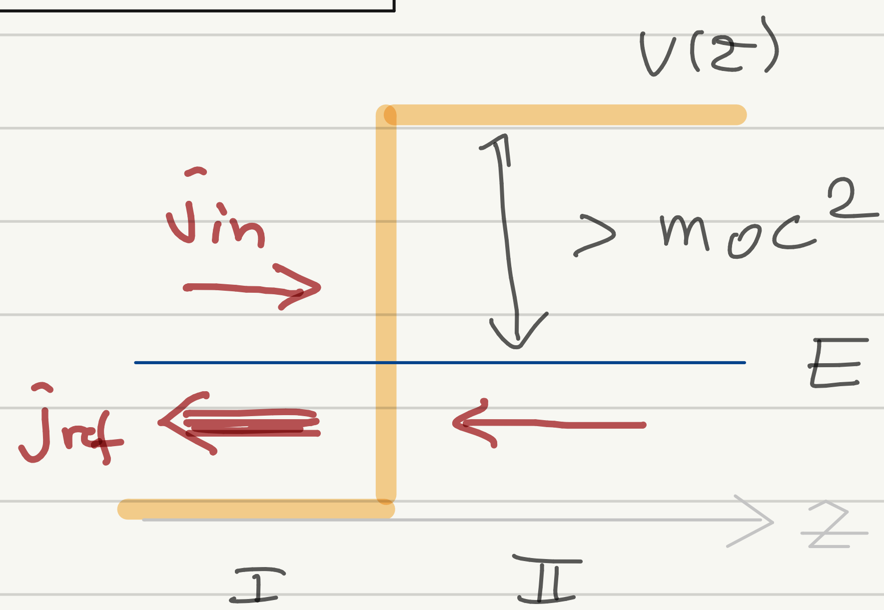
then

(561) $$\underline{\underline{R < 1}} \; !$$

transmission in classically forbidden regime ???

Klein paradoxon





In the problem course you will calculate the currents in the two cases and will show that for

$$V_0 > E + m_0 c^2$$

(562) $$|j_{ref}| > |j_{in}| \quad \text{and} \quad j_{II} < 0$$

<u>remark:</u> If the potential step gets smeared out over a distance d much larger than the compton length there is no current in region II

$$d \gg \lambda_c = \frac{\hbar}{m_0 c}$$