VII.2 Dirac equation and relativistic single-particle quantum mechanics of spin 1/2 particles

In the context of Dirac theory we can use the ordinary definitions of scalar product and expectation values

scalar product

$$\langle \psi | \psi' \rangle = \int d^3\vec{x}\; \psi^\dagger(\vec{x})\, \psi'(\vec{x})$$

(457)

$$= \int d^3\vec{x} \sum_{i=1}^{4} \psi_i^*(\vec{x})\, \psi_i'(\vec{x})$$

expectation value

(458) $$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle = \int d^3\vec{x}\; \psi^\dagger(\vec{x})\, \hat{O}_D\, \psi(\vec{x})$$

With this we can use the usual notions of

hermiticity & unitarity

As in the Klein-Gordon theory we have to ask ourselves however, if all hermitian operators allowed observables of a relativistic single-particle quantum mechanics?

True single-particle operators do not mix states of positive and negative frequency

<u>Def:</u> sign-operator

$$(459)\quad \hat{\Lambda} \equiv \frac{\hat{H}}{\sqrt{\hat{H}^2}} = \frac{c\,\vec{\alpha}\cdot\hat{\vec{p}} + \beta m_0 c^2}{c\sqrt{\vec{p}^{\,2} + m_0^2 c^2}}$$

$$\hat{\Lambda} = \hat{\Lambda}^{+} = \hat{\Lambda}^{-1} \qquad \text{eigenvalues: } \pm 1$$

<u>Def:</u> projection operators to positive and negative frequencies

$$(460)\qquad \boxed{\hat{\Pi}_{\pm} \equiv \frac{1}{2}\left(\mathbb{1} \pm \hat{\Lambda}\right)}$$

fulfills $\hat{\Pi}_{\pm}\,\Psi_{\pm} = \Psi_{\pm}$ $\qquad \hat{\Pi}_{\pm}\,\Psi_{\mp} = 0$

Every operator can be decomposed into an even and odd contribution. An even operator has only nonvanishing matrix elements between states of equal-sign frequencies, an odd operator only between states of unequal-sign frequencies.

i.e.

$$\boxed{\begin{aligned} \hat{A} &= [\hat{A}] + \{\hat{A}\} && (461) \\ \text{even} \quad [\hat{A}] &= \tfrac{1}{2}\left(\hat{A} + \hat{\Lambda}\hat{A}\hat{\Lambda}\right) && (462a) \\ \text{odd} \quad \{\hat{A}\} &= \tfrac{1}{2}\left(\hat{A} - \hat{\Lambda}\hat{A}\hat{\Lambda}\right) && (462b) \end{aligned}}$$

<u>example:</u>

- $\hat{H} = [\hat{H}]$ since $\hat{\Lambda}\hat{H}\hat{\Lambda} = \hat{H}$
- $\hat{\vec{p}} = [\hat{\vec{p}}]$
- $\left[\frac{d\hat{\vec{x}}}{dt}\right] \neq \frac{d\hat{\vec{x}}}{dt}$

since $\frac{d\hat{\vec{x}}}{dt} = -\frac{i}{\hbar}[\hat{\vec{x}}, \hat{H}] = -\frac{i}{\hbar}[\hat{\vec{x}}, c\hat{\vec{\alpha}}\cdot\hat{\vec{p}} + \beta m_0 c^2]$

$= c\vec{\alpha}$ eigenvalues $\pm c$

In a problem you will show that

$$\left[\frac{d\hat{\vec{x}}}{dt}\right] = c[\vec{\alpha}] = \frac{c\hat{\vec{p}}\hat{\Lambda}}{c\sqrt{\vec{p}^2 + m_0^2c^2}} \quad (463)$$

i.e the true velocity operator reads

positive frequencies

$$\frac{c\vec{p}}{E_p}$$

negative frequencies

$$-\frac{c\vec{p}}{E_p}$$

$\hookrightarrow$ negative kinematic mass

"Zitterbewegung" of a free electron

(464) $$\frac{d\vec{x}}{dt} = c\vec{\alpha} = \vec{v}$$

$$\frac{d\vec{v}}{dt} = \frac{1}{i\hbar} c\left[\vec{\alpha}, \hat{H}\right] = \frac{i}{\hbar} c\underbrace{\left(\vec{\alpha}\hat{H} + \hat{H}\vec{\alpha}\right)}_{\text{anti-commutator}} - \frac{2ic}{\hbar}\vec{\alpha}\hat{H}$$

$$\{\vec{\alpha}, \hat{H}\} = \{\vec{\alpha}, c\vec{\alpha}\cdot\hat{\vec{p}} + \beta m_0 c^2\}$$

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \qquad \{\alpha_j, \beta\} = 0$$

$\Rightarrow$

$$\frac{d\vec{v}}{dt} = \frac{2ic^2}{\hbar}\hat{\vec{p}} - \frac{2ic\vec{\alpha}\hat{H}}{\hbar} = \frac{2ic^2\hat{\vec{p}}}{\hbar} - \frac{2i\vec{v}\hat{H}}{\hbar}$$

since $\hat{\vec{p}}$ is a conserved quantity for a free particle the solution reads

$$(465)\quad \hat{\vec{v}}(t) = \left(\hat{\vec{v}}(0) - \frac{c^2\hat{\vec{p}}}{\hat{H}}\right) e^{\frac{-2i\hat{H}t}{\hbar}} + \frac{c^2\hat{\vec{p}}}{\hat{H}}$$

since

$$\frac{d\hat{\vec{v}}(t)}{dt} = \left(\hat{\vec{v}}(0) - \frac{c^2\hat{\vec{p}}}{\hat{H}}\right)\left(-\frac{2i\hat{H}}{\hbar}\right) e^{-\frac{2i\hat{H}t}{\hbar}}$$

$$= \left(\hat{\vec{v}}(t) - \frac{c^2\hat{\vec{p}}}{\hat{H}}\right)\left(-\frac{2i\hat{H}}{\hbar}\right)$$

$$= \frac{2ic\hat{\vec{p}}}{\hbar} - \frac{2i\hat{\vec{v}}(t)\hat{H}}{\hbar} \qquad \square$$

integration of (465) yields

$$(466)\quad \hat{\vec{x}}(t) = \hat{\vec{x}}(0) + \underbrace{\frac{c^2\hat{\vec{p}}}{\hat{H}}t}_{\text{like classical particle}} + \underbrace{\left(\hat{\vec{v}}(0) - \frac{c^2\hat{\vec{p}}}{\hat{H}}\right)\frac{i\hbar}{2\hat{H}t} e^{-\frac{2i\hat{H}t}{\hbar}}}_{\text{fast oscillation}}$$

amplitude $\sim \frac{\hbar}{2m_0c} = \frac{\lambda_c}{2}$

frequency $\sim \frac{2m_0c^2}{\hbar}$

"Zitterbewegung"

The oscillating part is an odd operator and only contributes if a wavepacket has both positive and negative frequency components!

$$(467) \quad \hat{\Pi}_\pm \left(\hat{\vec{v}}(0) - \frac{c^2\hat{\vec{p}}}{\hat{H}}\right)\frac{i\hbar}{2\hat{H}t} e^{-\frac{2i\hat{H}t}{\hbar}} \hat{\Pi}_\pm = 0$$

proof: $\quad [\hat{H}, \hat{\Pi}_\pm] = 0$ i.e.

$$\hat{\Pi}_\pm \left(\hat{\vec{v}}(0) - \frac{c^2\hat{\vec{p}}}{\hat{H}}\right)\frac{i\hbar}{2\hat{H}t} e^{-\frac{2i\hat{H}t}{\hbar}} \hat{\Pi}_\pm =$$

$$= \hat{\Pi}_\pm \left(\hat{\vec{v}}(0) - \frac{c^2\hat{\vec{p}}}{\hat{H}}\right)\hat{\Pi}_\pm \frac{i\hbar c}{\hat{H}} e^{-2i\hat{H}t/\hbar}$$

$$= \hat{\Pi}_\pm \left(\vec{\alpha}(0) - \frac{c\hat{\vec{p}}}{\hat{H}}\right)\hat{\Pi}_\pm \frac{i\hbar c^2}{\hat{H}} e^{-2i\hat{H}t/\hbar}$$

using $\quad [\hat{\Pi}_\pm, \vec{\alpha}] = \frac{1}{2}\left[(\mathbb{1} \pm \hat{\Lambda}), \vec{\alpha}\right] = \pm\frac{1}{2}[\hat{\Lambda}, \vec{\alpha}]$

$$= \pm\frac{1}{2}\left[\frac{\hat{H}}{E_p}, \vec{\alpha}\right]$$

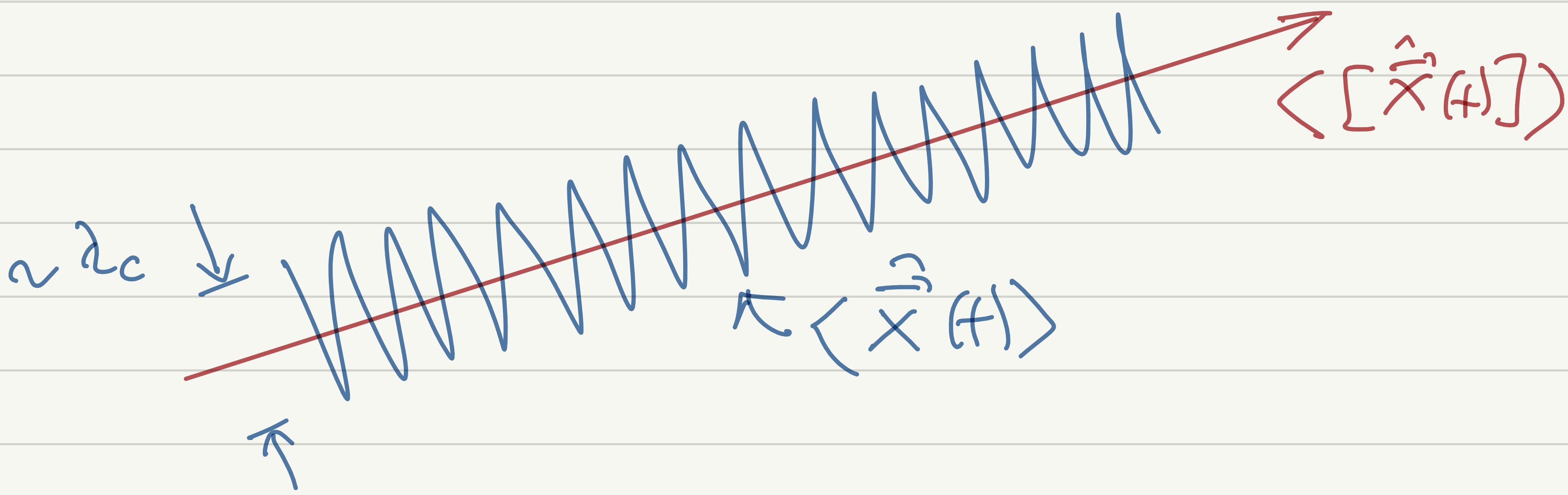
and using (464) i.e. $[\hat{H}, \vec{\alpha}] = 2c\hat{\vec{p}} - 2\vec{\alpha}\hat{H}$ gives

$$[\hat{\Pi}_\pm, \vec{\alpha}] = \pm \frac{c\hat{\vec{p}}}{E_p} \mp \frac{\vec{\alpha}\hat{H}}{E_p}$$

Thus $\quad \hat{\Pi}_\pm\left(\vec{\alpha} - \frac{c\hat{\vec{p}}}{\hat{H}}\right)\hat{\Pi}_\pm$

$$= \left(\vec{\alpha}\hat{\Pi}_\pm \pm \frac{c\hat{\vec{p}}}{E_p} \mp \frac{\vec{\alpha}\hat{H}}{E_p} - \frac{c\hat{\vec{p}}}{\hat{H}}\right)\hat{\Pi}_\pm$$

$$= \left(\vec{\alpha} \pm \frac{c\hat{\vec{p}}}{E_p} \mp \frac{\vec{\alpha}(\pm E_p)}{E_p} - \frac{c\hat{\vec{p}}}{\pm E_p}\right)\hat{\Pi}_\pm = 0$$



VII.3 Lorentz-invariance of Dirac theory; covariant formulation

Although the DE is first order in both time and space, we still have to prove that is invariant under Lorentz transformations. Also we have to discuss the transformation properties of the 4-component object Ψ, which will turn out <u>not</u> to be a Lorentz vector.

$$\Sigma_A \leftarrow x'_\mu = A_\mu^{\ \nu} x_\nu \longrightarrow \Sigma'_B$$

L-trafo

$$\psi(x) \qquad\qquad \psi'(x')$$

$$\left[i\hbar\partial_t - (c\vec{\alpha}\cdot\vec{p} + \beta m_0 c^2)\right]\psi(x) = 0$$

$$\left[i\hbar\partial_t' - (c\vec{\alpha}'\cdot\vec{p}' + \beta' m_0 c^2)\right]\psi'(x') \overset{?}{=} 0$$

Let us first introduce an alternative form of the Dirac equation

$$\frac{\beta}{c}\cdot\left(i\hbar\partial_t - (c\vec{\alpha}\cdot\vec{p} + \beta m_0 c^2)\right)\psi = 0$$

Def:

(468) $\boxed{\gamma^0 \equiv \beta}$ $\boxed{\gamma^i \equiv \beta\alpha_i}$ $i = 1,2,3$ (469)

γ - matrices

$\Rightarrow$

(470) $$i\hbar\left(\gamma^0\frac{\partial}{\partial x^0} + \gamma^1\frac{\partial}{\partial x^1} + \gamma^2\frac{\partial}{\partial x^2} + \gamma^3\frac{\partial}{\partial x^3}\right)\psi - m_0 c\psi = 0$$

The algebra of α_i, β matrices can be expressed in terms of γ matrices:

(471) $$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}}$$

Dirac algebra

explicit form

(472) $$\boxed{\gamma^0 = \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad \gamma^i = \beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}}$$

properties of γ matrices

4x4 matrices

(i) γ^μ is unitary

$$\boxed{(\gamma^\mu)^{-1} = (\gamma^\mu)^+} \quad (473)$$

(ii) γ^0 is hermitian

(474) $$\boxed{(\gamma^0)^+ = \gamma^0}$$

(iii) γ^i $(i=1,2,3)$ are anti-hermitian

(475) $$\boxed{(\gamma^i)^+ = -\gamma^i}$$

(Pauli) All representations of Dirac matrices are unitary equivalent.

Def.: Feynman dagger

(476)
$$\boxed{\begin{aligned} \not{B} &\equiv \gamma^\mu B_\mu = \gamma_\mu B^\mu = g_{\mu\nu}\gamma^\nu B^\mu \\ &= \gamma^0 B^0 - \sum_{i=1}^{3} \gamma^i B^i = \gamma^0 B^0 - \vec{\gamma}\cdot\vec{B} \end{aligned}}$$

in particular

(477)
$$\not{\partial} = \gamma^\mu \frac{\partial}{\partial x^\mu} = \frac{\gamma^0}{c}\frac{\partial}{\partial t} + \vec{\gamma}\cdot\vec{\nabla}$$

With this we can write the Dirac equation in the compact form

(478)
$$\boxed{\begin{aligned} (i\hbar\not{\partial} - m_0 c)\,\psi &= 0 \\ (\not{p} - m_0 c)\,\psi &= 0 \end{aligned}}$$

$$p^\mu = i\hbar\frac{\partial}{\partial x^\mu}$$

If there is a coupling to an external elm. field the DE reads

(479)
$$\boxed{\left(\not{p} - \frac{e\not{A}}{c} - m_0 c\right)\psi = 0}$$

We can rewrite the Hamilton operator in terms of γ-matrices

$$\hat{H} = c\vec{\alpha}\cdot\vec{p} + \beta m_0 c^2$$

(480)
$$= c\gamma^0\vec{\gamma}\cdot\vec{p} + \gamma^0 m_0 c^2$$

Lorentz invariance of Dirac equations means

(i) $\psi(x)$ in Σ_A must uniquely define $\psi'(x')$ in Σ_B with

$$x'_\mu = L_\mu{}^\nu x_\nu$$

and a linear transformation $\psi \to \psi'$

(ii) $\psi'(x')$ should fulfill DE (478) potentially with different Dirac matrices $\tilde{\gamma}^\mu$, but the $\tilde{\gamma}^\mu$ must be unitarily equivalent to γ^μ

What is transformation

$$\psi(x) \longrightarrow \psi'(x') \quad ?$$

ansatz for linear trafo ($L \mathrel{\hat{=}}$ Lorentz trafo)

(481)
$$\psi'(x') = \psi'(Lx) = S(L)\psi(x)$$
$$= S(L)\psi(L^{-1}x')$$

$S(L)$ is obviously a 4×4 matrix which must be invertable, i.e.

(482) $$\psi(x) = S^{-1}(L)\,\psi'(x') = S^{-1}(L)\,\psi'(Lx)$$

likewize it must hold for the inverse trafo

(483) $$\psi(x) = S(L^{-1})\,\psi'(Lx)$$

(484) so $$\boxed{S(L^{-1}) = S^{-1}(L)}$$

Since $\psi(x)$ shall obey the Dirac equation one has

$$\left[i\hbar\gamma^\mu S^{-1}(L)\frac{\partial}{\partial x^\mu} - m_0 c\, S^{-1}(L)\right]\psi'(x') = 0$$

multiplication with $S(L)$ from left yields

$$\left[i\hbar\, S(L)\gamma^\mu S^{-1}(L)\frac{\partial}{\partial x^\mu} - m_0 c\right]\psi'(x') = 0$$

since furthermore

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x^{\nu'}}{\partial x^\mu}\frac{\partial}{\partial x^{\nu'}} = L^{\nu'}_{\ \mu}\frac{\partial}{\partial x^{\nu'}}$$

$\Rightarrow$

(485) $$\left[i\hbar\, S(L)\gamma^\mu S^{-1}(L)\, L^{\nu'}_{\ \mu}\frac{\partial}{\partial x^{\nu'}} - m_0 c\right]\psi'(x') = 0$$

In order for eq. (485) to be equivalent to the Dirac equation for $\psi'(x')$, $S(L)$ must fulfill

(486) $$S(L)\gamma^{\mu}S^{-1}(L)L^{\nu'}_{\ \mu} = \gamma^{\nu'}$$

or

(487) $$\boxed{L^{\nu'}_{\ \mu}\gamma^{\mu} = S^{-1}(L)\gamma^{\nu'}S(L)}$$

Def.: A 4-component wavefunction is called a Lorentz 4-spinor or bi-spinor if it transforms under Lorentz trafo

$$x^{\mu} \longrightarrow x^{\nu'} = L^{\nu'}_{\ \mu}x^{\mu}$$

according to

(488) $$\boxed{\psi'(x') = S(L)\,\psi(L\cdot x)}$$

where
$$L^{\nu'}_{\ \mu}\gamma^{\mu} = S^{-1}(L)\gamma^{\nu'}S(L)$$

$\Rightarrow$ Lorentz invariance of Dirac theory

(A) Construction of Spinortransformation under proper Lorentz transformations

Let us consider first proper L-trafo's, i.e. without spatial or time inversion. They can be generated continuously out of the identical map by infinitesimal transformations.

▶ infinitesimal Lorentz trafo

(489) $$L^{\nu'}_{\ \mu} = \delta^{\nu'}_{\mu} + \delta\omega^{\nu'}_{\ \mu}$$

a) boost in x-direction

(490) $$\delta\omega^1_{\ 0} = \delta\omega^0_{\ 1} = -\frac{\delta v}{c}$$

$$x^{0'} = x^0 - \frac{\delta v}{c} x^1 \qquad x^{2'} = x^2$$

$$x^{1'} = x^1 - \frac{\delta v}{c} x^0 \qquad x^{3'} = x^3$$

i.e. $$\delta\omega_{10} = g_{11}\delta\omega^1_{\ 0} = -\delta\omega_{01} = -g_{00}\delta\omega^0_{\ 1}$$

(491) $$\delta\omega_{10} = -\delta\omega_{01} \qquad \delta\omega^{10} = -\delta\omega^{01}$$

b) rotation about z-axis

(492) $$\delta\omega^1_{\ 2} = -\delta\omega^2_{\ 1} = \delta\varphi$$

$$x^{0'} = x^0 \qquad x^{1'} = x^1 + \delta\varphi\, x^2$$

$$x^{3'} = x^3 \qquad x^{2'} = x^2 - \delta\varphi\, x^1$$

i.e. $\delta\omega^{12} = -\delta\omega^1{}_2 = \delta\omega^2{}_1 = -\delta\omega^{21}$

(493) $\underline{\underline{\delta\omega_{12} = -\delta\omega_{21} \qquad \delta\omega^{12} = -\delta\omega^{12}}}$

generalizing this to arbitrary directions

(494)
$$\boxed{\begin{aligned} &\delta\omega^{\mu\nu} = -\delta\omega^{\mu\nu} \qquad \mu,\nu = 0,1,2,3 \\ &\delta\omega^{\mu\mu} = 0 \end{aligned}}$$

- What is $S(\delta^\mu{}_\nu + \delta\omega^\mu{}_\nu) = \,?$

Expansion in powers of $\delta\omega^\mu{}_\nu$:

(495)
$$S = \mathbb{1}_4 - \frac{i}{4}\sigma_{\mu\nu}\,\delta\omega^{\mu\nu} \equiv \mathbb{1} + \tau$$
$$S^{-1} = \mathbb{1}_4 + \frac{i}{4}\sigma_{\mu\nu}\,\delta\omega^{\mu\nu} \equiv \mathbb{1} - \tau$$

Note that each component $\sigma_{\mu\nu}$ is a 4×4 matrix!
Since $\delta\omega^{\mu\nu}$ is antisymmetric $\Rightarrow$

(496) $\boxed{\sigma_{\mu\nu} = -\sigma_{\nu\mu}}$ $\Rightarrow 6$ components

Insertion of (495) into condition (487) yields

$$(\mathbb{1}-\tau)\gamma^\nu(1+\tau) = \gamma^\nu + \gamma^\nu\tau - \tau\gamma^\nu$$
$$\stackrel{!}{=} (\delta^\nu_\mu + \delta\omega^\nu_\mu)\gamma^\mu$$
$$= \gamma^\nu + \delta\omega^\nu_\mu \gamma^\mu$$

This gives

(497) $$[\gamma^\nu, \tau] = \delta\omega^\nu_\mu \gamma^\mu$$

In order to conserve the norm of the bispinor we can further require

(498) $$\det(S) = 1$$

$$\det(S) = \det(\mathbb{1}+\tau) = 1 + \mathrm{Tr}\{\tau\}$$

(499) $$\Rightarrow \quad \mathrm{Tr}\{\tau\} = 0$$

With this we find for the infinitesimal Spinortrafo

(500)
$$\boxed{\begin{gathered} S = \mathbb{1}+\tau \qquad \tau = -\frac{i}{4}\delta\omega^{\mu\nu}\sigma_{\mu\nu} \\ \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] = \begin{cases} 0 & \mu=\nu \\ -i\alpha^j & \mu=0,\ \nu=j \\ \varepsilon_{ijk}\Sigma^k & \mu=i,\ \nu=j \end{cases} \\ \Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \end{gathered}}$$

(501) or $$S = \mathbb{1} + \frac{1}{8}[\gamma_\mu, \gamma_\nu]\,\delta\omega^{\mu\nu}$$

The proper infinitesimal L-Transformation can in general be decomposed into boosts $\delta\eta_j = \frac{\delta v_i}{c}$ and rotations $\delta\varphi_j$ along respectively about the $j = 1,2,3$ axis

(502) $$\boxed{\delta\omega = i\,\delta\varphi_j I^j - i\,\delta\eta_j K^j}$$

where $\delta\omega$ is a 4x4 matrix and the matrices I^j and K^j

rotation

(503) $$I^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad I^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad I^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

boost

(504) $$K^1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K^2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K^3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

the matrices fulfill

(505) $$\boxed{[I^i, I^j] = i\,\varepsilon_{ijk} I^k}$$

(506) $$\boxed{[K^i, K^j] = -i\,\varepsilon_{ijk} I^k}$$

i.e. rotation are a subgroup of the proper Lorentz group
boosts are not

▶ finite Lorentztrafo

To obtain the spinortransformation of a finite L-Trafo we make use of the fact that a finite Trafo is just a sequence of infinitesimal trafos

rotation about axis j

$$L \longrightarrow R_j(\varphi) = \lim_{N\to\infty} R_j^N\left(\frac{\varphi}{N}\right)$$

$$= \lim_{N\to\infty}\left(1 + i\frac{\varphi}{N}I^j\right)^N = \exp\left(i\varphi I^j\right)$$

$$\boxed{R_j(\varphi) = e^{i\varphi I^j}} \tag{507}$$

boost along axis j

$$L \longrightarrow B_j\left(\frac{v}{c}\right) = \lim_{N\to\infty} B_j^N\left(\frac{v}{cN}\right)$$

$$= \lim_{N\to\infty}\left(1 - i\frac{v}{cN}K^j\right)^N = \exp\left(-i\frac{v}{c}K^j\right)$$

(508) $$\boxed{B_j\left(\frac{v}{c}\right) = e^{-i\frac{v}{c}K^j}}$$

Finite Spinortrafo

We note that from the commutation relations (505) and (506) follows

(509) $$\boxed{\begin{aligned} \left[S(I^i), S(I^j)\right] &= 2i\varepsilon_{ijk} S(I^k) \\ \left[S(K^i), S(K^j)\right] &= -2i\varepsilon_{ijk} S(I^k) \\ \left[S(I^i), S(K^j)\right] &= 2i\varepsilon_{ijk} S(K^k) \end{aligned}}$$

Finite spinortransformations can then be obtained by exponentiating infinitesimal trafos as done above for the L-Trafo. This yields

rotation

(510) $$\boxed{\begin{aligned} S(R_j) &= e^{i\frac{\varphi}{2}S(I^j)} = e^{i\frac{\varphi}{2}\Sigma^j} \\ &= \cos\frac{\varphi}{2} + i\Sigma^j \sin\frac{\varphi}{2} \end{aligned}}$$

boost

(511) $$\boxed{\begin{aligned} S(B_j) &= e^{-i\frac{v}{2c}S(K^j)} = e^{-\frac{v}{2c}\alpha^j} \\ &= \cosh\frac{v}{2c} + \alpha^j \sinh\frac{v}{2c} \end{aligned}}$$

Rem.: Note the factor $\frac{1}{2}$ in front of $\varphi \Sigma^j$ in eq. (510)

Bispinor $\psi(x)$ goes over to itself under rotation of 4π (not 2π)!

e.g.

$$\psi'(x') = S(R_z(\varphi))\, \psi(R_z(\varphi)x)$$

$$= e^{i\frac{\varphi}{2}\Sigma^z}\, \psi(R_z(\varphi)x)$$

$\varphi = 2\pi$ $\qquad \psi'(x') = e^{i\pi\Sigma^z}\, \psi(x)$

$\varphi = 4\pi$ $\qquad \psi'(x') = \psi(x)$

(B) Spinortransformation under space and time inversion

Let us now discuss what happens under the special Lorentztransformations

space inversion

(512)
$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

time reversal

(513)
$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $S(P)$ and $S(T)$ as well as $S(PT)$ have to fulfill the condition of Lorentzinvariance (487) follows

(514)
$$S^{-1}(P)\gamma^0 S(P) = \gamma^0 \qquad\qquad [S(P),\gamma^0]=0$$
$$\Rightarrow$$
$$S^{-1}(P)\gamma^j S(P) = -\gamma^j \qquad\qquad \{S(P),\gamma^0\}=0$$

(515) $\Rightarrow$
$$\boxed{S(P) = e^{i\varphi_P}\gamma^0} \qquad \varphi_P \text{ arbitrary}$$

for $S(PT)$ where $PT = TP = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ one finds

(516)
$$S^{-1}(PT)\gamma^0 S(PT) = -\gamma^0 \qquad\qquad \{S(PT),\gamma^0\}=0$$
$$S^{-1}(PT)\gamma^j S(PT) = \gamma^j \qquad\qquad [S(PT),\gamma^j]=0$$

(517) $\Rightarrow$
$$\boxed{S(PT) = e^{i\varphi_{PT}}\gamma^5} \qquad \varphi_{PT} \text{ arbitrary}$$

where

(518)
$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3}$$

Now using $S(T) = S(P)S(PT)$ we find

(519)
$$\boxed{S(T) = e^{i\varphi_T}\gamma^0\gamma^5}$$

dropping the phase terms we obtain

(520)
$$S(P) = \gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad S(T) = \gamma^0\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$
$$S(PT) = \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

(C) Adjoint Spinor transformation

To see how quantities like $\psi^+ A \cdot \psi$ behave under Lorentz trafo we need to investigate

$$\psi^+ \longrightarrow \psi^+ S^+$$

We see that for rotations

$$\Sigma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}$$

(521)
$$S(R_j)^+ = \left(e^{i\frac{\varphi}{2}\Sigma^j}\right)^+ = e^{-i\frac{\varphi}{2}\Sigma^j} = S(R_j)^{-1}$$

i.e. for rotations S is unitary! However for boosts

(522)
$$S(B_j)^+ = \left(e^{-\frac{v}{2c}\alpha^j}\right)^+ = e^{-\frac{v}{2c}\alpha^j} = S(B_j) \neq S^{-1}(B_j)$$

not unitary!

However it holds

(523) $$\boxed{\gamma^0 S^\dagger \gamma^0 = S^{-1}}$$

This means objects like $\psi^\dagger \psi$ are <u>not</u> invariant under Lorentz-Trafo

<u>Def:</u> adjoint bi-spinor

(524) $$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

The quantity $\bar{\psi}\psi$ is now a true Lorentz scalar

$$\bar{\psi}'(x')\psi'(x') = \psi'^\dagger(x')\gamma^0\psi'(x')$$

(525) $$= \psi^\dagger(x) S^\dagger \gamma^0 S \psi(x) = \psi^\dagger(x) \underbrace{\gamma^0\gamma^0}_{=1} S^\dagger \gamma^0 S \psi$$

$$= \bar{\psi}(x) S^{-1} S \psi(x)$$

$$= \bar{\psi}(x)\psi(x) \qquad \square$$

Lets have a look at the 4-current density

$$j^\mu = (c\psi^\dagger\psi;\ c\psi^\dagger\vec{\alpha}\psi)$$

this can be written as

(526) $$j^\mu = c\,\psi^\dagger \gamma^0 \gamma^\mu \psi = c\,\bar{\psi}\gamma^\mu \psi$$

How does j^μ transform under Lorentz-trafo's?

$$j^{\mu'}(x') = c\,\psi'^\dagger(x')\gamma^0\gamma^\mu\psi'(x')$$
$$= c\,\psi^\dagger(x) S^\dagger \gamma^0 \gamma^\mu S \psi(x)$$
$$= c\,\psi^\dagger(x)\gamma^0 S^{-1}\gamma^\mu S\psi(x)$$

Due to the Lorentz invariance condition (487)

$$L^\mu_{\ \nu}\gamma^\nu = S^{-1}\gamma^\mu S$$

we obtain

$$\underline{\underline{j^{\mu'}(x')}} = c\,\psi^\dagger(x)\gamma^0 L^\mu_{\ \nu}\gamma^\nu\psi(x)$$

(527) $$= L^\mu_{\ \nu}\; c\,\psi^\dagger(x)\gamma^0\gamma^\nu\psi(x)$$

$$= \underline{\underline{L^\mu_{\ \nu}\, j^\nu(x)}}$$

i.e. the 4-current density is a Lorentz vector

(528) $$\boxed{j^\mu = c\,\bar{\psi}\gamma^\mu\psi}$$

(D) Bilinear covariant expressions of Dirac bispinors

We have already encountered two bilinear expressions of Dirac spinors that transform as a Lorentz tensor

$\bar{\psi}\psi$ scalar

$j^\mu = c\bar{\psi}\gamma^\mu\psi$ vector

Both have the structure: adjoint spinor γ-matrix spinor

Out of the γ matrices we can build 16 linear independent 4x4 matrices (dimension $4^2=16$)

(529)

$$\boxed{\begin{array}{llll} \text{scalar} & \mathbb{1} & \text{vector} & \gamma^\mu \\ \text{tensor} & \sigma^{\mu\nu} & \text{pseudoscalar} & \gamma^5 \\ \text{pseudovector} & \gamma^5\gamma^\mu & & \end{array}}$$

proper L-trafo (rotation, boost) $\quad S\gamma_5 S^{-1} = \gamma_5 \qquad \gamma_5 = \gamma^5$

space inversion $\quad P\gamma_5 P^{-1} = -\gamma_5$

bilinear forms

(530)	(i) scalar	$\bar{\psi}\psi$
	(ii) pseudoscalar	$\bar{\psi}\gamma_5\psi$
	(iii) vector	$\bar{\psi}\gamma^\mu\psi$
	(iv) pseudovector	$\bar{\psi}\gamma_5\gamma^\mu\psi$
	(v) tensor rank 2	$\bar{\psi}\sigma^{\mu\nu}\psi$

VII.4 Solution of Dirac equation for some simple problems

(A) Plane waves generated by Lorentz trafo

For a Dirac particle at rest we have

(531) $$\psi(\vec{r},t) = \omega^r(0)\, e^{-i\varepsilon_r m_0 c^2 t/\hbar}$$

$$r = 1,2,3,4$$

$$\varepsilon_{1,2} = +1 \qquad \varepsilon_{3,4} = -1$$