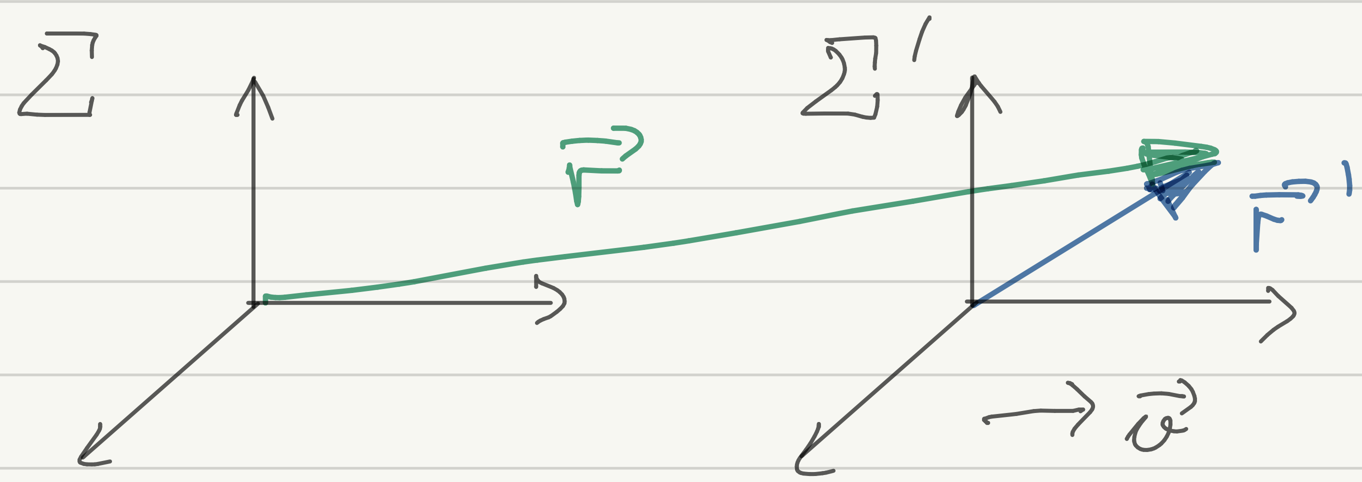
V. Special theory of relativity: a short summary

In the remainder of this lecture we will discuss a possible generalization of quantum mechanics to a Lorentz invariant form. To this end let us first summarize the key points of the special theory of relativity (SRT)

V.1 Gallilei- and Lorentz trafo; postulates of SRT

The general laws of physics should be the same in all inertial frames



(268) $$\boxed{\vec{r}\,' = \vec{r} - \vec{v}t, \qquad t' = t}$$ Gallilei Trafo

- Newton's equations are invariant under Gallilei transformations.

- let us have a look what happens with Maxwell's equations

e.g.: wave equation of electric field in vacuum

(269) in Σ_1: $$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{r},t) = 0$$

in Σ_1':

$$\frac{\partial}{\partial x_j} = \frac{\partial x_j'}{\partial x_j}\frac{\partial}{\partial x_j'} \overset{(268)}{=} \frac{\partial}{\partial x_j'}$$

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x_i'}{\partial t}\frac{\partial}{\partial x_i'} = \frac{\partial}{\partial t'} - v_i\frac{\partial}{\partial x_i'}$$

i.e.

(270) $$\Big[\Delta' - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} + \underbrace{\frac{2}{c^2}\vec{v}\cdot\vec{\nabla}'\frac{\partial}{\partial t'} - \frac{1}{c^2}(\vec{v}\cdot\vec{\nabla}')(\vec{v}\cdot\vec{\nabla}')}_{\text{not invariant!}}\Big]\vec{E} = 0$$

$\Rightarrow$

(i) Maxwell-equations not Gallilei invariant!

two possible consequences:

a) there exists a special coordinate frame ("ether") and MWE in other reference frames look different

b) Gallilei transformation is not the correct transformation between inertial frames

experiments of Michelson, Morley (1881, 1887) Fizeau (1851)

$\Rightarrow$ • vacuum (!) speed of light is independent of inertial frame

• concept of "ether" is not resonable

postulates of SRT (Einstein, Poincare)

I) laws of nature have identical form in all inertial frames

II) vacuum speed of light is independent of the inertial frame

But then the Gallilei-transformation cannot be generally correct!

In the limit of small relative velocities the Gallilei trafo must be correct, however.

$\Rightarrow$ search for correct transformation time-evolution of wave-front in time dt (Σ_1)

$$c^2 dt^2 - |d\vec{r}|^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) = 0$$

2nd postulate: also in Σ_1'

$$c^2 dt'^2 - (dx'^2 + dy'^2 + dz'^2) = 0$$

def: $x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z$

assume $\vec{v} = v\hat{e}_x = v\hat{e}_{x'}$

$$dx^{0\prime 2} - dx^{1\prime 2} = dx^{0\,2} - dx^{1\,2} \qquad dx^{2\prime} = dx^2 \quad dx^{3\prime} = dx^3$$

- transformation is linear
- inverse transformation: $\vec{v} \to -\vec{v}$

↳ Lorentz transformation

(271)
$$\boxed{\begin{array}{ll} x^{0\prime} = \gamma (x^0 - \beta x^1) & x^{2\prime} = x^2 \\ x^{1\prime} = \gamma (x^1 - \beta x^0) & x^{3\prime} = x^3 \end{array}}$$
L-Trafo

(272)
$$\boxed{\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}}$$

(273) $$\boxed{\begin{array}{ll} x^0 = \gamma(x^{0'} + \beta x^{1'}) & x^{2'} = x^2 \\ x^1 = \gamma(x^{1'} + \beta x^{0'}) & x^{3'} = x^3 \end{array}}$$ Inverse L-Trafo

for $\left|\frac{v}{c}\right| \ll 1$: $\beta x^{1'} = \frac{v}{c} x \to 0$ $\beta x^{0'} = \frac{v}{c} ct \to vt$

$$x^{0'} = x^0$$
$$x^{1'} = x^1 - vt$$
$\Rightarrow$ Gallilei

L-Trafo mixes spatial and time coordinates; i.e. it is useful to introduce a 4-dimensional space:

V.2 4-vectors and Minkowski space

(A) 4-vectors and metric tensor

(274) $$(x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

vector in 4-dimensional space

(275) $$\underline{x} = x^0 \vec{e}_0 + x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = x^\mu \vec{e}_\mu \; {}^*$$

Einstein convention: sum over pairwise indices from 0 to 3

* notation with upper and lower indices will become clear later

Scalar product

(276) $$\underline{x}\cdot\underline{y} = x^\mu y^\mu = ?$$

this turns out to be not a good definition as we would expect that a Lorentz-trafo should not change the length of a vector

$$\sqrt{\underline{x}\cdot\underline{x}} = \|\underline{x}\|$$

With (276) it would do this however

$$\underline{x} = (ct, x, 0, 0)$$

$$\underline{x}' = (\gamma ct - \gamma\beta x, \gamma x - \gamma\beta ct, 0, 0)$$

$$\|\underline{x}\| = \sqrt{c^2t^2 + x^2}$$

$$\|\underline{x}'\| = \sqrt{\gamma^2(ct-\beta x)^2 + \gamma^2(x-\beta ct)^2}$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Big[c^2t^2 + \frac{v^2}{c^2}x^2 - 2vtx + x^2 + v^2t^2 - 2vtx\Big]^{1/2}$$

$$\neq \|\underline{x}\|$$ ☹

better

(277) $$\boxed{\underline{x}\cdot\underline{y} = \sum_{\mu,\nu} g_{\mu\nu}x^\mu x^\nu = g_{\mu\nu}x^\mu x^\nu}$$

where

(278) $$\boxed{g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}$$ metric tensor

infinitesimal length of vector

(279) $$d\underline{s} = \vec{e}_\mu \, dx^\mu$$

where (280) $$\boxed{\vec{e}_\mu \cdot \vec{e}_\nu = g_{\mu\nu}}$$

$$d\underline{s} \cdot d\underline{s} = ds^2 = (dx^0)^2 - \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right)$$
$$= c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

This was invariant under Lorentz trafo!

4-vectors with (280) + (290) $\Rightarrow$ Minkowski space

Due to the minus signs in $g_{\mu\nu}$ the scalar product of a vector with itself is no longer positiv, which sounds a bit strange on first glance but is very useful.

A length (difference between two points in the 4-dim. space)

(281)	$\Delta s^2 = \Delta \underline{s} \cdot \Delta \underline{s} = 0$	light-like distance
	$\Delta s^2 > 0$	time-like distance
	$\Delta s^2 < 0$	space-like distance

(B) covariant and contravariant coordinates

orthogonal basis vectors

(282) $$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

Here we have seen that does not hold; i.e. the basis vectors behave not as "usual orthogonal" vectors.

- what happens if we consider a skewed coordinate system

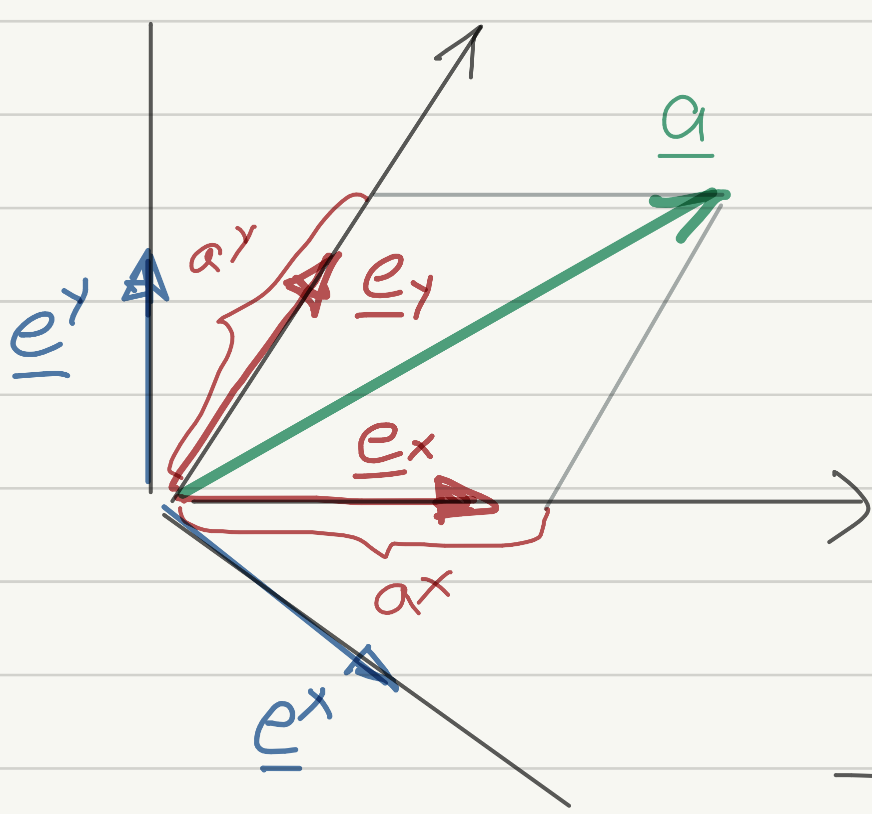
$$\underline{a} = a^x \underline{e}_x + a^y \underline{e}_y \quad (283)$$

but $\underline{e}_x \cdot \underline{e}_y \neq 0$

introduce new basis

$$\underline{e}^x, \underline{e}^y \quad (284)$$

$$\underline{e}_x \cdot \underline{e}^y = 0 \quad (285)$$



more general

(286) $$\boxed{\underline{e}_i \cdot \underline{e}^j = \delta_{ij}}$$

so we can represent a vector $\underline{a}$ in two different ways

(287)
$$\underline{a} = a_i \underline{e}^i \qquad a_i \text{ - covariant component of } \underline{a}$$

$$\underline{a} = a^i \underline{e}_i \qquad a^i \text{ - contravariant component of } \underline{a}$$

- How to relate the two representations?

(288) $$\boxed{\underline{e}_i = \sum_j g_{ij} \underline{e}^j}$$

proof: $$\underline{e}_i \cdot \underline{e}_j = \sum_l g_{il} \underbrace{\underline{e}^l \cdot \underline{e}_j}_{\delta_{jl}} = g_{ij} \quad \square$$

likewise

(289) $$\boxed{\underline{e}^i = \sum_j g^{ij} \underline{e}_j} \qquad \boxed{g^{ij} \equiv g_{ij}}$$

scalar product of two vectors $\underline{a}, \underline{b}$

$$\underline{a}\cdot\underline{b} = a_j \underline{e}^j \cdot b^l \underline{e}_l = a_j b^l \underbrace{\underline{e}^j\cdot\underline{e}_l}_{=\delta_{jl}}$$

$$= a_j b^j$$

$$= a^j \underline{e}_j \cdot b_l \underline{e}^l = a^j b_l \underbrace{\underline{e}_j\cdot\underline{e}^l}_{=\delta_{jl}}$$

$$= a^j b_j$$

(2.90) $$\boxed{\underline{a}\cdot\underline{b} = a_j b^j = a^j b_j}$$

(C) Lorentztrafo as pseudo rotation

(2.91) $$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \cosh\varphi$$

(2.92) $$\gamma\beta = \frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \sinh\varphi$$

Note that

$$\cosh^2\varphi - \sinh^2\varphi = \frac{1}{1-\frac{v^2}{c^2}} - \frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} = 1$$

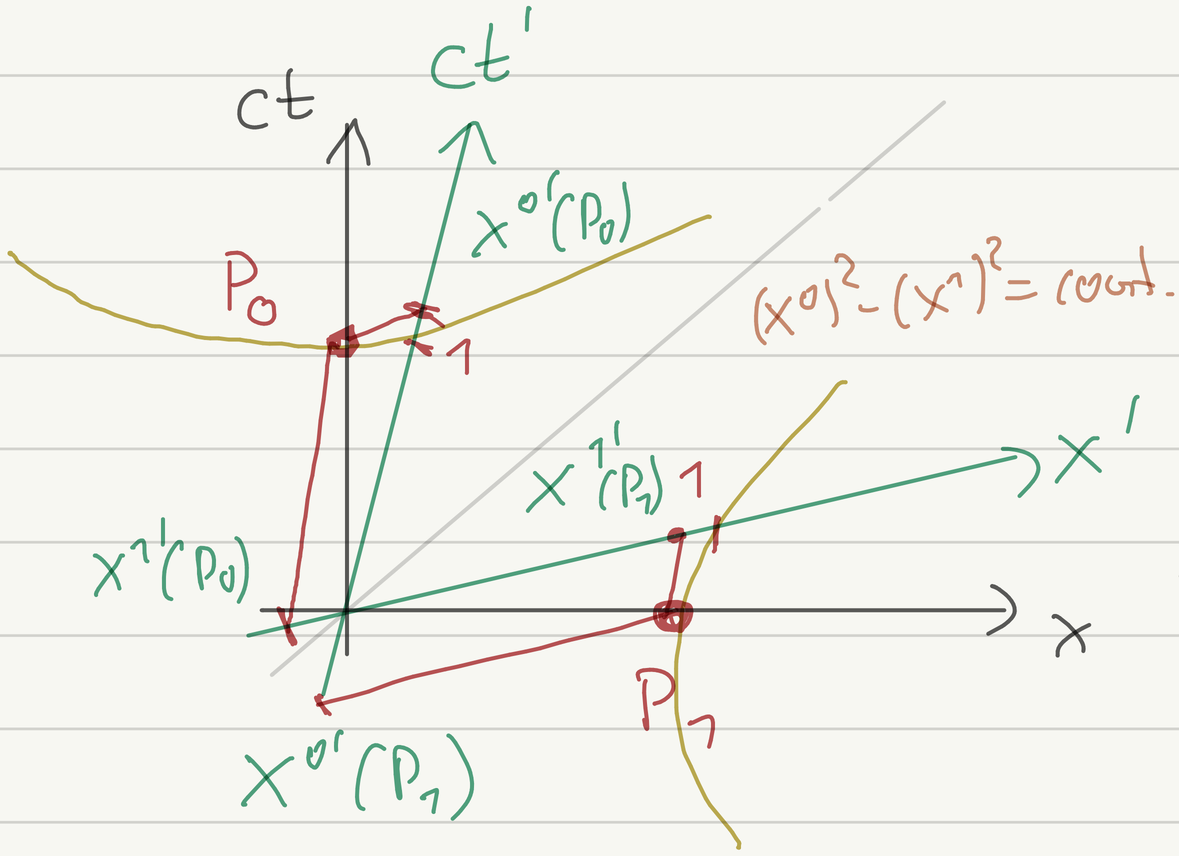
i.e.

$$x^{0\prime} = \cosh\varphi\, x^0 - \sinh\varphi\, x^1$$

$$x^{1\prime} = -\sinh\varphi\, x^0 + \cosh\varphi\, x^1$$

(2.93)

for $\vec{v} \parallel \vec{e}_x$



P_0:

$x^0 = 1 \qquad x^1 = 0$

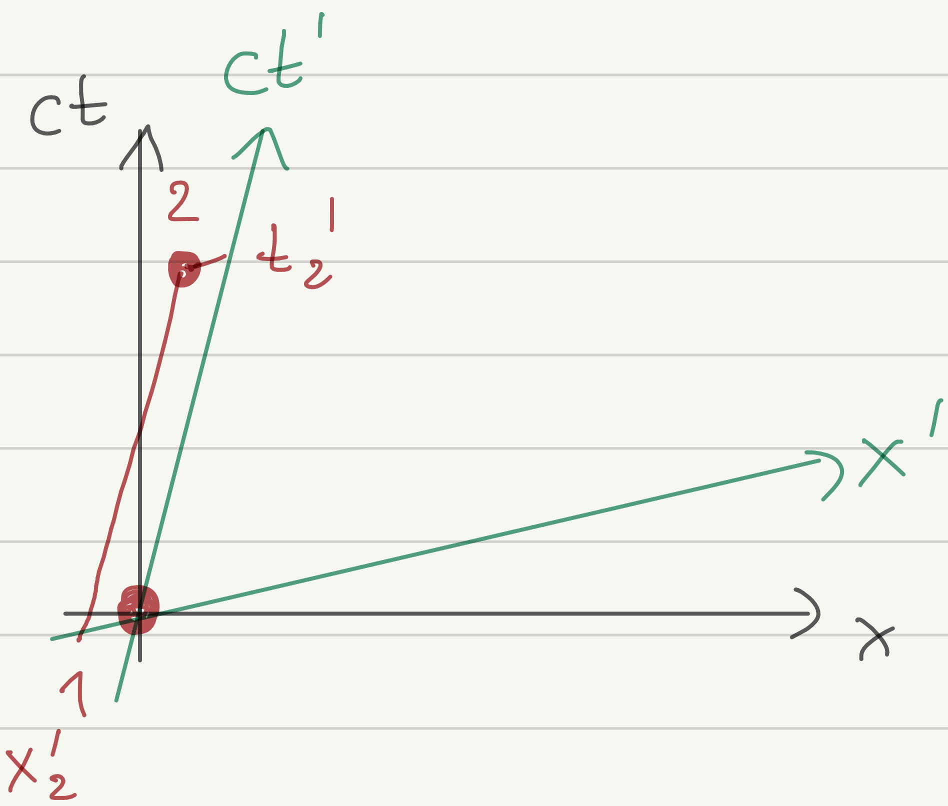
$x^{0'} = \cosh\varphi \qquad x^{1'} = \sinh\varphi$

P_1:

$x^0 = 0 \qquad x^1 = 1$

$x^{0'} = -\sinh\varphi \qquad x^{1'} = \cosh\varphi$

time-like distance

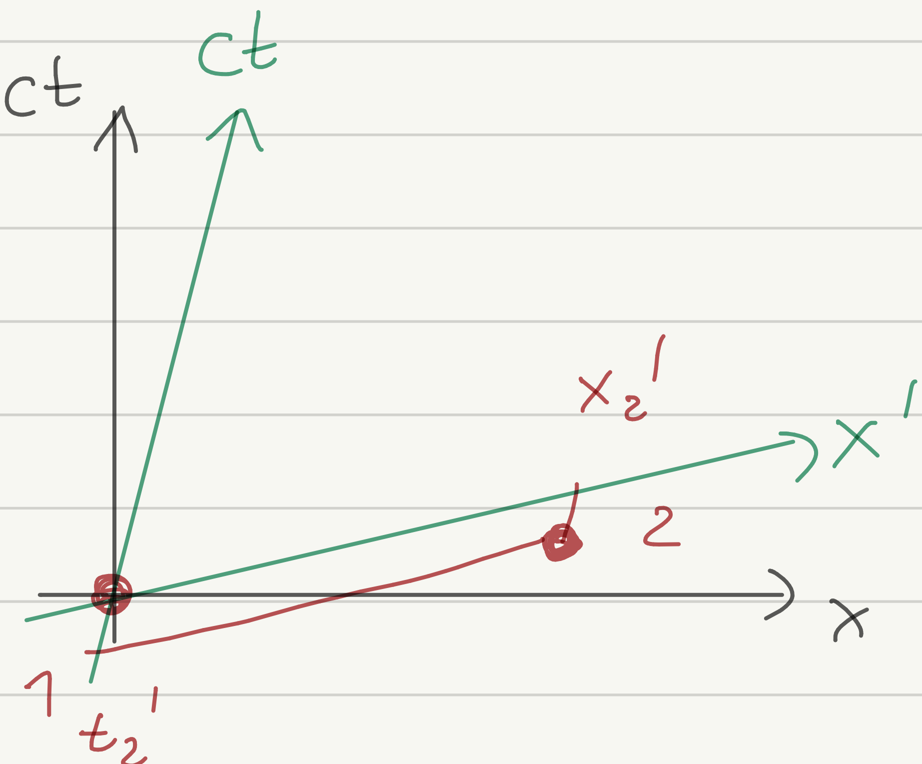


$t_2 > t_1 \qquad t_2' > t_1'$

$x_2 > x_1 \qquad x_2' < x_1'$

$\boxed{\text{time order conserved, spatial order not}}$

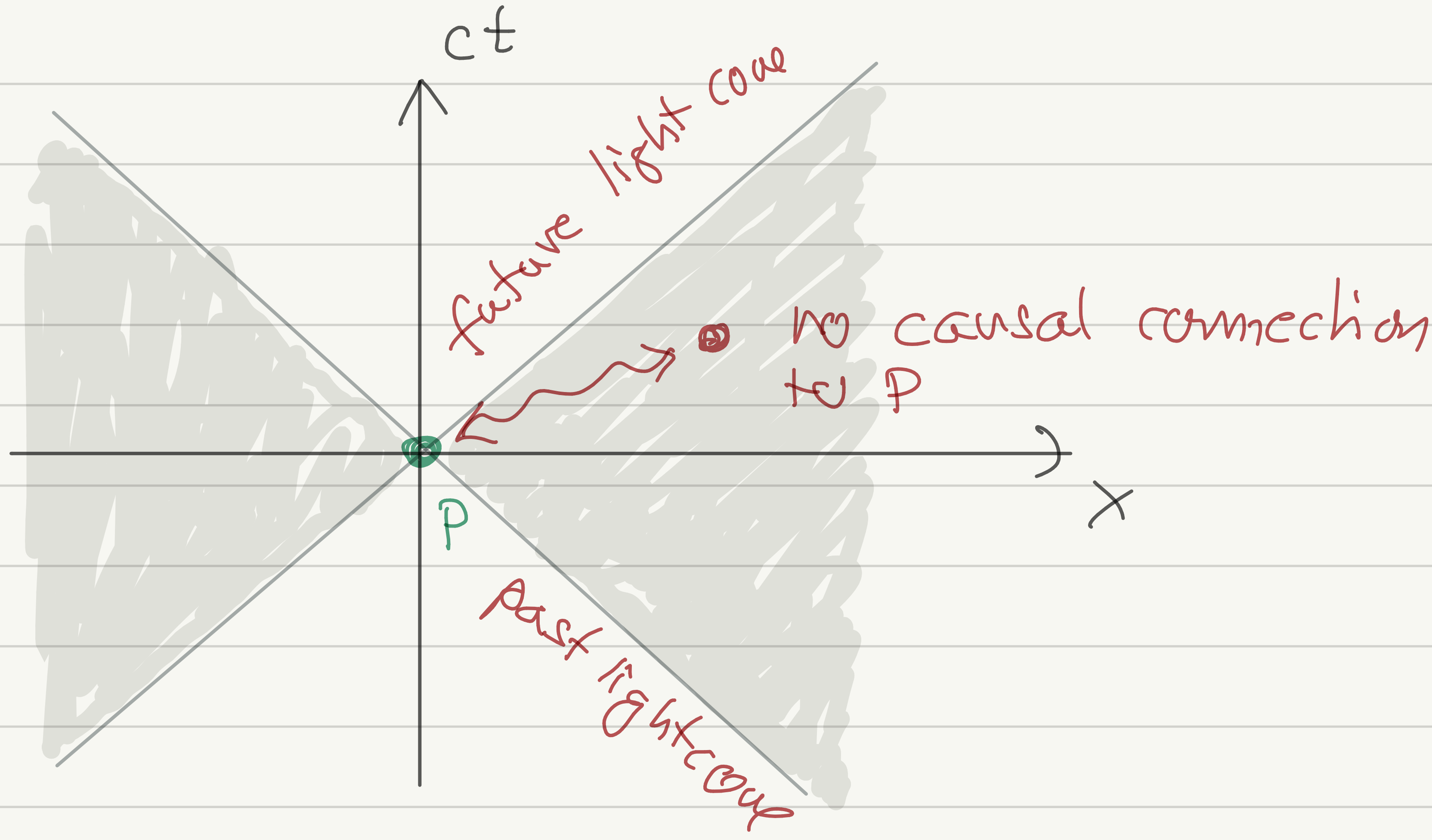
space-like distance



$t_2 > t_1 \qquad t_2' < t_1'$

$x_2 > x_1 \qquad x_2' > x_1'$

$\boxed{\text{spatial order conserved, time order not}}$



(D) Geometric objects and tensors

Def: a geometric object $T^{ijk\ldots}_{\ \ \ \ \ lmn\ldots}$ is called

tensor if it transforms under a coordinate transformation as the coordinates

coordinate trafo: $\Sigma \rightarrow \Sigma'$

$$dx^{j'} = A^{j'}_{\ \ l}\, dx^{l}$$

(294) $$\boxed{T^{i'j'k'\ldots}_{\ \ \ \ \ \ \ l'm'n'\ldots} = A^{i'}_{\ \ i} A^{j'}_{\ \ j} A^{k'}_{\ \ k} \cdots A^{l}_{\ \ l'} A^{m}_{\ \ m'} A^{n}_{\ \ n'}\, T^{ijk}_{\ \ \ \ lmn}}$$

Def: it is called pseudo-tensor if

(295) $$\boxed{T^{i'j'\ldots}_{\ \ \ \ \ l'm'\ldots} = \mathrm{sgn}(A^{i'}_{\ \ i})\, A^{i'}_{\ \ i} A^{j'}_{\ \ j} \cdots A^{l}_{\ \ l'} A^{m}_{\ \ m'}\, T^{ij\ldots}_{\ \ \ \ lm\ldots}}$$

Any physical law that can be formulated as a relation of Lorentz tensors of the same kind will have the same form in all inertial frames i.e. will fulfill the 2nd postulate of SRT.

V.3 Lorentz invariance of Edynamics

Let us show that Maxwells Edynamics can be formulated as relations between Lorentz tensors in Minkowski space

MWE in vacuum (Gauß units)

(296)
$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{j}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = 0$$

continuity equation of charge

(297)
$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0$$

elm. potentials

(298)
$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi$$

Lorentz gauge

(299) $$\frac{1}{c}\frac{\partial}{\partial t}\phi + \vec{\nabla}\cdot\vec{A} = 0$$

$\Rightarrow$

(300) $$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{j}$$
$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho$$

from (300) we see that we can introduce a

4-potential

(301) $$\boxed{\begin{aligned} A^\mu &\equiv (\phi, \vec{A}) \\ A_\mu &\equiv (\phi, -\vec{A}) \end{aligned}}$$

and a

4-current density

(302) $$\boxed{\begin{aligned} j^\mu &\equiv (c\rho, \vec{j}) \\ j_\mu &\equiv (c\rho, -\vec{j}) \end{aligned}}$$

4-coordinate and 4-gradient

$$x^\mu = (ct, \vec{r})$$
$$x_\mu = (ct, -\vec{r})$$

$$\boxed{\begin{aligned} \partial_\mu &= \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) \\ \partial^\mu &= \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) \end{aligned}}$$ (303)

wave-operator

(304)
$$\boxed{\Box \equiv \partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta}$$

with these we can formulate the Edynamics in L-invariant form

Lorentz gauge
(305)
$$\boxed{\partial^\mu A_\mu = \partial_\mu A^\mu = \frac{1}{c}\frac{\partial A^0}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0}$$

wave eqs.
(306)
$$\boxed{\Box A^\mu = \frac{4\pi}{c} j^\mu}$$

continuity eq.
(307)
$$\boxed{\partial^\mu j_\mu = \partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$$

electrodynamics is Lorentz invariant

The formulation in terms of potentials is technically the most easy way to show the L-invariance of Edyn. It has the disadvantage that one has to choose a specific gauge, the Lorentz gauge.

One can find a gauge-invariant formulation in terms of electric and magnetic fields

$$\vec{E}, \vec{B} \quad \rightarrow \quad 6 \text{ objects !?}$$

field (strength) tensor

$$\boxed{F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha} \tag{308}$$

is antisymmetric $\Rightarrow$ 6 independent components ☺

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

inhomogeneous MWE

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta} \tag{309}$$

dual field tensor

$$\boxed{\mathcal{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}} \tag{310}$$

$\varepsilon^{0123} = 1$
+ cyclic
+ anti-symmetric

homogeneous MWE

$$\boxed{\partial_\alpha \mathcal{F}^{\alpha\beta} = 0} \tag{311}$$

V.4 Relativistic Mechanics

- Newton mechanics $\vec{r}, \frac{d\vec{r}}{dt}$ or $\vec{p} = m\frac{d\vec{r}}{dt}$, $E = \frac{\vec{p}^2}{2m}$
- relativistic mechanics

$$\vec{r} \rightarrow x^\mu = (ct, \vec{r})$$

problem: $\dot{x}^\mu = (c, \dot{\vec{r}})$ not Lorentzvector ☹

proper time

(312) $$dt \rightarrow \boxed{d\tau = dt\sqrt{1-\beta^2} = \frac{1}{\gamma} dt}$$

since
$$ds^2 = c^2dt^2 - |d\vec{r}|^2 = c^2dt^2\left(1-\frac{v^2}{c^2}\right) = c^2d\tau^2$$
$\Rightarrow d\tau$ Lorentz invariant!

4-momentum

(313) $$\boxed{p^\mu = m_0\frac{dx^\mu}{d\tau} = m_0\gamma\frac{dx^\mu}{dt} = \gamma\left(m_0c, m_0\frac{d\vec{r}}{dt}\right)}$$

kinetic energy in non-relativistic mechanics $E = \frac{\vec{p}^2}{2m_0}$

relativistic energy and mass

(314) $$\boxed{E = mc^2 = m_0 \gamma c^2 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} c^2}$$

(315) $$\boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}$$

i.e.

(316) $$\boxed{p^\mu = \left(\frac{E}{c}, \vec{p}\right)}$$ where $\vec{p} = m\frac{d\vec{r}}{dt}$

Note that here "m" and not "m_0" appears!

$$p_0^2 - \vec{p}\cdot\vec{p} = p^\mu p_\mu = m_0 c^2$$

$$E^2 = p_0^2 c^2 = c^2 p^2 + m_0^2 c^4$$

Or

(317) $$\boxed{E = \sqrt{c^2 p^2 + m_0^2 c^4}}$$ energy of a free relativistic particle