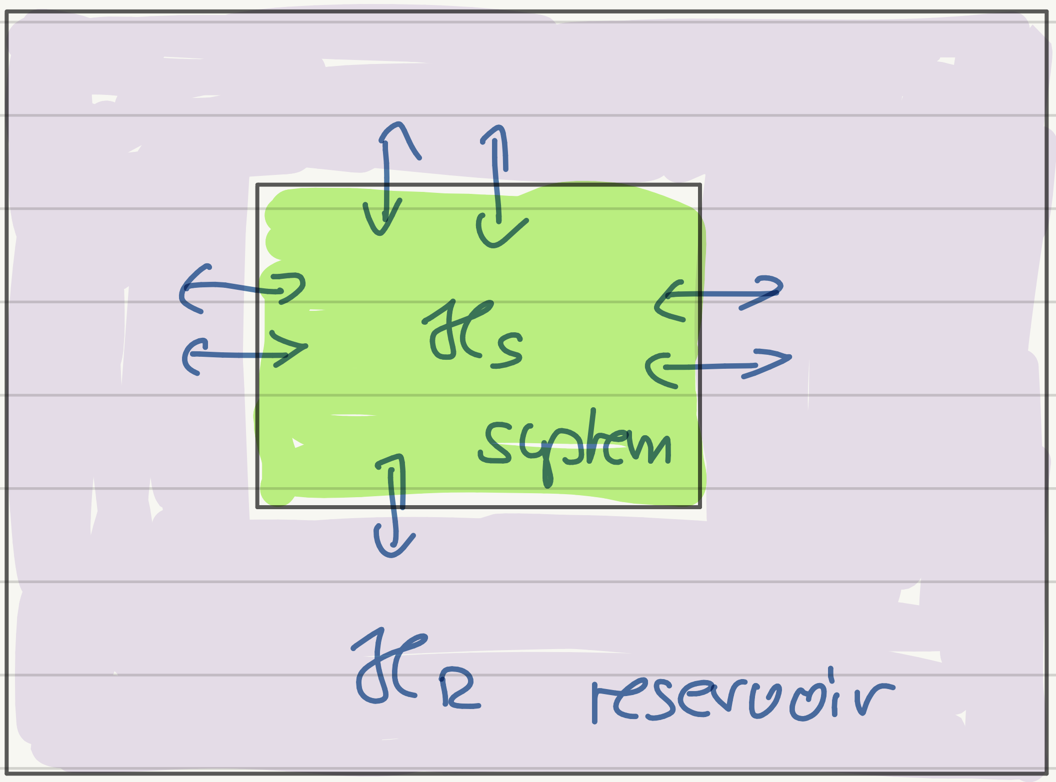
IV. Quantum mechanics of open systems

A common problem in quantum physics is that of a small system S coupled to a much larger one called reservoir R where we are interested however only in observables of the small system S



$\dim(\mathcal{H}_S) \ll \dim(\mathcal{H}_R)$

Can we find an effective description of the small system without the uninteresting degrees of freedom of the reservoir?

For certain types of reservoirs, so-called Markovian reservoirs an elegant formalism exists:

Lindblad equations

This formalism cannot be applied to all situations in quantum physics but applies well e.g. to quantum-optical systems and quantum gases. An important different approach is the Caldeira-Legget approach not covered in this class.

effective description of S only means that we loose information and we can no longer describe the system with a state in Hilbert space

IV.1 Mixed states & density matrix*

(* a reminder)

Suppose we have a quantum system for which we only know that it is in certain q.m. state $|\phi_n\rangle$ with probability p_n. How to describe this in quantum physics?

(193) $$|\psi\rangle = \sum_n \sqrt{p_n}\,|\phi_n\rangle \quad ?$$

then for observable

$$\langle \hat{A}\rangle = \sum_{n,m}{}' \sqrt{p_n p_m}\,\langle \phi_n|\hat{A}|\phi_m\rangle \quad \lightning$$

it should rather be

(194) $$\langle \hat{A}\rangle = \sum_n p_n \langle \phi_n|\hat{A}|\phi_n\rangle$$

$\Rightarrow$ state vector no longer appropriate!

However: all information about q. system is in the projector

(195) $$\varrho_{|\psi\rangle} \equiv |\psi\rangle\langle\psi|$$

since $$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

(196) $$= \mathrm{Tr}\left\{ |\psi\rangle\langle\psi|\hat{A} \right\}$$

$\varrho_{|\psi\rangle}$ fulfills the equation of motion

$$i\hbar \frac{d}{dt}\varrho_{|\psi\rangle} = |\mathring{\psi}\rangle\langle\psi| + |\psi\rangle\langle\mathring{\psi}|$$

$$= H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|H$$

$$= \left[H, \varrho_{|\psi\rangle}\right]$$

(197) $$\boxed{\mathring{\varrho}_{|\psi\rangle} = -\frac{i}{\hbar}\left[H, \varrho_{|\psi\rangle}\right]}$$ Schrödinger eq. for projector on state

The projector description can be generalized to systems with reduced information (probabilities)

(198) $$|\phi_n\rangle \longrightarrow \boxed{\varrho = \sum_n p_n |\phi_n\rangle\langle\phi_n|}$$

state vector — mixed state, density operator

$0 \leq p_n \leq 1 \quad \sum_n p_n = 1$ probabilities observables

$$\langle \hat{A} \rangle = \langle \phi_n | \hat{A} | \phi_n \rangle$$

$$\Downarrow$$

(199) $$\boxed{\begin{aligned} \langle \hat{A} \rangle = \mathrm{Tr}\{\rho \hat{A}\} &= \sum_n p_n \langle \phi_n | \hat{A} | \phi_n \rangle \\ &= \sum_n p_n \langle \hat{A} \rangle_n \end{aligned}}$$

properties of density operator

(i) $\boxed{\rho^\dagger = \rho}$ self adjoint (200)

$\Rightarrow$ real eigenvalues

(ii) $\forall |\psi\rangle \in \mathcal{H}_s$ (201) $\boxed{\langle \psi | \rho | \psi \rangle \geq 0}$ positive

$$\langle \psi | \rho | \psi \rangle = \sum_n p_n \langle \psi | \phi_n \rangle \langle \phi_n | \psi \rangle = \sum_n p_n |\langle \psi | \phi_n \rangle|^2 \geq 0$$

eigenvalues of ρ are probabilities

(iii) a non-trivial but important property is

$\forall |\phi\rangle \in \mathcal{H}_s \otimes \mathcal{H}_{aux}$ $\forall \rho_{aux}$ auxiliary system

(202) $\boxed{\langle \phi | \rho \otimes \rho_{aux} | \phi \rangle \geq 0}$ complete positivity

(iv) from $\sum_n p_n = 1$ follows

(203) $$\boxed{Tr\{\rho\} = \sum_n p_n = 1}$$ conservation of probability

i.e. $\frac{d}{dt} Tr\{\rho\} = 0$

(v) (204) $$\boxed{Tr\{\rho^2\} \leq Tr\{\rho\} = 1}$$

$$Tr\{\rho^2\} = \sum_n \langle\psi_n| \underbrace{\sum_{m,k} p_m p_k |\psi_m\rangle}_{\sigma_{mn}} \underbrace{\langle\psi_m|\psi_k\rangle}_{\delta_{mk}} \langle\psi_k|\psi_n\rangle$$

$$= \sum_n p_n^2 \leq \sum_n p_n = Tr\{\rho\} = 1$$

(205) $$\boxed{Tr\{\rho^2\} = 1 \iff \rho = |\psi\rangle\langle\psi|}$$ pure state

(vi) If ρ_1 and ρ_2 are density matrices then also

(206) $$\boxed{\rho = p\rho_1 + (1-p)\rho_2}$$ is convexity

convex sum

(vii) If all eigenvalues $0 \leq p_n \leq 1$ of ρ are different there is a unique diagonal representation of ρ

(207) $$\boxed{\rho = \sum_n p_n |\phi_n\rangle\langle\phi_n|}$$ $p_n \neq p_m$ $\forall n,m$

IV.2 Time evolution of a subsystem in the Markov limit: Lindblad equation

We now want to develop an effective description of the dynamics of the small system coupled to a reservoir in the so-called Markov limit of a reservoir "without memory"

(208) $$\hat{H} = \underbrace{\hat{H}_S}_{\text{system}} + \underbrace{\hat{H}_R}_{\text{reservoir}} + \underbrace{\hat{H}_{SR}}_{\text{system-reservoir coupling}}$$

Liouville equation of total state

(209) $$\chi \in \mathcal{H}_S \otimes \mathcal{H}_R \qquad \dot{\chi} = -\frac{i}{\hbar}\left[H, \chi\right]$$

If we only want to calculate observables $\hat{A}$ that act in the Hilbert space of the system, then

$$\langle \hat{A} \rangle = \mathrm{Tr}_{S+R}\{\chi \hat{A}\}$$
$$= \mathrm{Tr}_S\{\mathrm{Tr}_R\{\chi\}\hat{A}\}$$

it is sufficient to consider the reduced density matrix

(210) $$\rho \equiv \mathrm{Tr}_R\{\chi\}$$

ϱ is in general a mixed state even if χ was a pure state. This is the case if and only if system and reservoir are in an entangled state.

$$\chi = |\psi\rangle\langle\psi|$$

Using a Schmidt decomposition $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j}\, |j\rangle_S |j\rangle_R \tag{2.11}$$

where $\{|j\rangle_S\}$ and $\{|j\rangle_R\}$ are ONB in $\mathcal{H}_S$ and $\mathcal{H}_R$ respectively and $d \leq \min\{\dim(\mathcal{H}_S), \dim(\mathcal{H}_R)\}$

The λ_j are called Schmidt coefficients. If only one coefficient is non-zero the state $|\psi\rangle$ is factorized if more than one value $\lambda_j \neq 0$ the state $|\psi\rangle$ is entangled between S and R. Then

$$\varrho = \mathrm{Tr}_R\{|\psi\rangle\langle\psi|\} = \sum_{j=1}^{d} \lambda_j\, |j\rangle_{SS}\langle j| \tag{2.12}$$

Since

$$\mathrm{Tr}\{\varrho\} \overset{!}{=} 1 \Rightarrow \sum_{j=1}^{d} \lambda_j \overset{!}{=} 1 \tag{2.13}$$

but

$$\mathrm{Tr}\{\varrho^2\} = \sum_{j=1}^{d} \lambda_j^2 \leq 1 \tag{2.14}$$

equality holds if only one $\lambda_j = 1$ is nonzero

What is the dynamics of S?

It is useful to split-off the fastest timescales associated with the free dynamics $H_S + H_R$ by a unitary trafo

(215a) $$\chi(t) \to \tilde{\chi}(t) = e^{\frac{i}{\hbar}(H_S+H_R)t}\, \chi(t)\, e^{-\frac{i}{\hbar}(H_S+H_R)t}$$

and

(215b) $$A \to \tilde{A}(t) = e^{\frac{i}{\hbar}(H_S+H_R)t}\, A\, e^{-\frac{i}{\hbar}(H_S+H_R)t}$$

such that $\mathrm{Tr}\{\chi(t)A\} = \mathrm{Tr}\{\tilde{\chi}(t)\tilde{A}(t)\}$

Then

(216) $$\dot{\tilde{\chi}}(t) = -\frac{i}{\hbar}\left[\tilde{H}_{SR}(t), \tilde{\chi}(t)\right]$$

Let us assume a weak coupling between reservoir and system (Born approximation) $\Rightarrow$ time dependent perturbation theory. First integrate eq (216)

(217) $$\tilde{\chi}(t) = \tilde{\chi}(t_0) - \frac{i}{\hbar}\int_{t_0}^{t} d\tau\, \left[\tilde{H}_{SR}(\tau), \tilde{\chi}(\tau)\right]$$

substitute (217) into r.h.s. of eq. (216)

(218) $$\dot{\tilde{\chi}}(t) = -\frac{i}{\hbar}\left[\tilde{H}_{SR}(t), \tilde{\chi}(t_0)\right] - \frac{1}{\hbar^2}\int_{t_0}^{t} d\tau\, \left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(\tau), \tilde{\chi}(\tau)\right]\right]$$

Now we assume that t_0 was chosen such that system and reservoir were still factorized (e.g. $t_0 \to -\infty$), then

(219) $$\tilde{\chi}(t_0) = \rho_S(t_0) \otimes \rho_R(t_0)$$

Now we note that the 2nd term on the r.h.s. of (218) contains H_{SR} to second order. In the sense of time-dependent perturbation we can approximate in the term on the r.h.s.

(220) $$\tilde{\chi}(\tau) \simeq \tilde{\rho}(\tau) \otimes \tilde{\rho}_R(\tau)$$ Born approximation

Furthermore we assume that the reservoir is in an equilibrium state, i.e.

(221) $$\left[\tilde{H}_R(\tau), \tilde{\rho}_R(\tau)\right] = 0$$

i.e.

$$\tilde{\rho}_R(\tau) = \tilde{\rho}_R(t_0) = \rho_R$$

Now we can trace eq. (218) over the reservoir degrees of freedom

(222) $$\dot{\tilde{\rho}}(t) = \underbrace{-\frac{i}{\hbar} Tr_R \left\{ \left[\tilde{H}_{SR}(t), \tilde{\rho}(t_0) \otimes \rho_R \right] \right\}}_{\text{often zero} \to \text{ignore}}$$

$$- \frac{1}{\hbar^2} \int_{t_0}^{t} d\tau \, Tr_R \left\{ \left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(\tau), \tilde{\rho}(\tau) \otimes \rho_R \right] \right] \right\}$$

Let us assume a generic reservoir-system interaction of the type

(223) $$H_{SR} = \hbar g \sum_j \hat{S}_j \hat{R}_j$$

where $\hat{S}_j$ and $\hat{R}_j$ are system or reservoir operators respectively.

here $\hat{S}_j = \hat{S}_j^+$ $\hat{R}_j = \hat{R}_j^+$ generalization possible

in interaction picture (215) but dropping the tilde for ease of notation:

$$\tilde{X}(t) \to X(t)$$

we have

(224) $$\dot{\rho}(t) = -\sum_{jk} g^2 \int_{t_0}^{t} d\tau \, Tr_R \left\{ \left[\hat{S}_j(t)\hat{R}_j(t), \left[\hat{S}_k(\tau)\hat{R}_k(\tau), \rho(\tau)\rho_R \right] \right] \right\}$$

$$= -g^2 \sum_{jk} \int_0^t d\tau \left\{ \langle \hat{R}_j(t)\hat{R}_k(\tau) \rangle_R \left[\hat{S}_j(t)\hat{S}_k(\tau)\rho(\tau) - \hat{S}_k(\tau)\rho(\tau)\hat{S}_j(t) \right] \right.$$
$$\left. - \langle \hat{R}_j(\tau)\hat{R}_k(t) \rangle_R \left[\hat{S}_j(t)\rho(\tau)\hat{S}_k(\tau) - \rho(\tau)\hat{S}_k(\tau)\hat{S}_j(t) \right] \right\}$$

where

(225) $$\langle \hat{R}_j(t)\hat{R}_k(\tau) \rangle_R \equiv Tr_R \{ \rho_R \hat{R}_j(t) \hat{R}_k(\tau) \}$$

Eq. (224) is a closed equation for the reduced density matrix ρ but it is non-local in time.

Since the reservoir is in an equilibrium state, two-time correlations are stationary, i.e.

$$(226) \qquad \langle \hat{R}_j(t)\hat{R}_k(\tau)\rangle = R_{jk}(t-\tau) \xrightarrow[|t-\tau|\to\infty]{} 0$$

If the reservoir correlations (226) decay on a time scale which is much faster than the dynamics in the system we can make the so-called

Markov approximation

$$(227) \qquad \boxed{\begin{aligned} g^2\langle \hat{R}_j(t)\hat{R}_k(\tau)\rangle &\simeq (\Gamma_{jk} + i\Delta_{jk})\,\delta(t-\tau) \\ g^2\langle \hat{R}_j(\tau)\hat{R}_k(t)\rangle &\simeq (\Gamma_{jk} - i\Delta_{jk})\,\delta(t-\tau) \\ \text{where} \quad \Gamma_{jk} &= \Gamma_{kj} \qquad \Delta_{jk} = \Delta_{kj} \end{aligned}}$$

This approximation is usually well justified in quantum optics and atomic physics. In condensed-matter systems it is often a bad approximation. (227) yields

$$(228) \qquad \begin{aligned} \dot{\rho} = &-\frac{1}{2}\sum_{jk} \gamma_{jk}\left\{\hat{S}_j\hat{S}_k\rho - \hat{S}_k\rho\hat{S}_j\right\} \\ &-\frac{1}{2}\sum_{jk} \gamma_{jk}^{*}\left\{\hat{S}_j\rho\hat{S}_k - \rho\hat{S}_k\hat{S}_j\right\} \end{aligned}$$

this can be written as

(229)
$$\dot{\rho} = -\frac{i}{2}\sum_{jk} \Delta_{jk} \left[\hat{S}_j \hat{S}_k , \rho\right]$$
$$+ \frac{1}{2}\sum_{jk} \Gamma_{jk} \left\{ 2\hat{S}_j \rho \hat{S}_k - \hat{S}_k \hat{S}_j \rho - \rho \hat{S}_k \hat{S}_j \right\}$$

correction to system Hamiltonian

Since $\Gamma_{jk} = \Gamma_{kj}$ is symmetric it can be diagonalized.

Furthermore if we generalize the discussion to non-hermitian operators $\hat{S}_j$ and $\hat{R}_j$ one eventually arrives at the

Lindblad-Masterequation

(230)
$$\boxed{\begin{aligned}\dot{\rho} &= -\frac{i}{\hbar}\left[H_{ef}, \rho\right] \\ &+ \frac{1}{2}\sum_{\mu} \Gamma_{\mu} \left\{ 2\hat{L}_{\mu} \rho \hat{L}_{\mu}^{\dagger} - \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rho - \rho \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \right\}\end{aligned}}$$

One can show that

(231)
$$\Gamma_{\mu} \geq 0$$

The $\hat{L}_{\mu}$ are called Lindblad-operators or jump operators

Some properties of Lindblad equation

- The Lindblad equation conserves probability as

(232) $$\frac{d}{dt} \mathrm{Tr}\{\rho\} = 0$$

- It does not conserve the purity

(233) $$\frac{d}{dt} \mathrm{Tr}\{\rho^2\} \neq 0$$

I.e. open-system dynamics can lead to increased mixedness but also to a purification of the state.

- It also conserves the (complete) positivity of $\rho(t)$

- Eq. (230) generates a non-unitary time evolution corresponding to a semi-group (inverse time evolution is not element)

- Eq. (230) is invariant under unitary transformations

(234) $$\sqrt{\Gamma_\mu}\,\hat{L}_\mu \to \sqrt{\Gamma_\mu'}\,\hat{L}_\mu' = \sum_\nu U_{\mu\nu} \sqrt{\Gamma_\nu}\,\hat{L}_\nu$$

- under

(235) $$\hat{L}_\mu \to \hat{L}_\mu' = \hat{L}_\mu + c_\mu \qquad c_\mu \in \mathbb{C}$$

$$\hat{=} \quad H \to H' = H + \frac{1}{2i} \sum_\mu \Gamma_\mu \left(c_\mu^* \hat{L}_\mu - c_\mu \hat{L}_\mu^\dagger\right) + \text{const}$$

(235) allows to make all $\hat{L}_\mu$ traceless

- Liouville super-operator (linear map

(236) $$\dot{\varrho} = \mathcal{L}\varrho$$

$$\mathcal{L}\cdot = -\frac{i}{\hbar}[H,\cdot] + \frac{1}{2}\sum_\mu \Gamma_\mu \left(L_\mu \cdot L_\mu^+ - \cdot L_\mu^+ L_\mu - L_\mu^+ L_\mu \cdot\right)$$

- $\mathcal{L}$ has right eigenvalues and eigenvectors

(237) $$\boxed{\mathcal{L}\varrho_\lambda = -\lambda\varrho_\lambda \qquad \lambda \neq \lambda^*}$$

where

(238) $$\boxed{Re[\lambda] \geq 0}$$ are decay rates

ϱ can be decomposed into ϱ_λ

(239) $$\boxed{\varrho(t) = \sum_\lambda c_\lambda \varrho_\lambda e^{-\lambda t}}$$

Since $Tr\{\varrho(t)\} = \text{const}_t \Rightarrow$

(240) $$Tr\{\varrho_\lambda\} \equiv 0 \quad \text{for } \lambda \neq 0$$

i.e. the ϱ_λ are themselves no density matrices!

The density matrix for $\lambda=0$ is called stationary state; it is a density matrix

(241) $$\rho_{ss} = \rho_0 = \lim_{t\to\infty} \rho(t)$$

where it was assumed that no other eigenstate ρ_λ with $\mathrm{Re}[\lambda]=0$ exists.

Complex eigenvalues come in conjugate pairs as

$$\rho(t) = \rho^*(t)$$

(242) $$\sum_\lambda c_\lambda \rho_\lambda e^{-\lambda t} \overset{!}{=} \sum_\lambda c_\lambda^* \rho_\lambda^* e^{-\lambda^* t} \quad \forall t$$

IV.3 Examples of dissipative systems

(A) harmonic oscillator coupled to reservoir of oscillators

(243) $$H_S = \hbar\omega\left(\hat{a}^+\hat{a}+\tfrac{1}{2}\right) \qquad H_R = \sum \hbar\omega_k\left(\hat{b}_k^+\hat{b}_k+\tfrac{1}{2}\right)$$

(244) $$H_{SR} = \hbar\sum_k g_k\left(\hat{a}^+\hat{b}_k + \hat{a}\hat{b}_k^+\right)$$

b_k-oscillators shall be in a finite-temperature state

(245) $$\rho_R = \prod_k e^{-\beta\hbar\omega_k b_k^+ b_k}\left(1-e^{-\beta\hbar\omega_k}\right)$$

in a rotating frame

$$\tilde{a} = a e^{-i\omega t} \qquad \tilde{b}_k = b_k e^{-i\omega_k t}$$

short notation

$$B = \sum_k g_k \hat{b}_k \qquad B^+ = \sum_k g_k b_k^+$$

(246) $$\tilde{B} = \sum_k g_k \hat{b}_k e^{-i\omega_k t}$$

now we identify $\tilde{a} \to \tilde{S}_1 \quad \tilde{B}^+ \to \tilde{R}_1$

$\tilde{a}^+ \to \tilde{S}_2 \quad \tilde{B} \to \tilde{R}_2$

i.e.

(247) $$H_{SR} = S_1 R_1 + S_2 R_2 \qquad (\text{w.o. tilde})$$

Since

$$\mathrm{Tr}\{\rho_R \hat{b}_k \hat{b}_e\} = \mathrm{Tr}\{\rho_R \hat{b}_k^+ \hat{b}_e^+\} = 0$$

$\Rightarrow$

(248) $$\langle R_1(t) R_1(\tau)\rangle = \langle R_2(t) R_2(\tau)\rangle = 0$$

but

$$\langle R_1(t) R_2(\tau)\rangle = \sum_{ke} g_k g_e e^{+i\omega_k t} e^{-i\omega_e \tau} \underbrace{\langle b_k^+ b_e\rangle}_{= \delta_{ke} \bar{n}_k}$$

(249) $$= \sum_k g_k^2 e^{i\omega_k (t-\tau)} \bar{n}_k$$

(250) $$\langle R_2(t) R_1(\tau)\rangle = \sum_k g_k^2 e^{-i\omega_k (t-\tau)} (\bar{n}_k + 1)$$

so

$$\underbrace{\langle R_1(t) R_2(\tau)\rangle} \ldots \tilde{a}(t) \ldots \tilde{a}^+(\tau) \ldots$$

(257) $$\sim \sum_k g_k^2 \underbrace{e^{+i(\omega_k-\omega)(t-\tau)}}_{\text{only for } \omega_k \approx \omega \text{ relevant}} \bar{n}_k \quad \Rightarrow \bar{n}_k \simeq \bar{n}(\omega)$$

$$\underset{\text{Markov approximation}}{\cong} (\gamma + i\Delta)\, \bar{n}(\omega)\, \delta(t-\tau)$$

We see that for the Markov approximation to hold

$$g_k^2 n_k$$

should vary slowly in ω_k as compared to the characteristic inverse time of the system evolution

$\Rightarrow$ spectrally broad reservoir!

since then $$\sum_k g_k^2 \bar{n}_k e^{+i(\omega_k-\omega)(t-\tau)}$$

$$\simeq \bar{g}^2 \bar{n}(\omega) \underbrace{\int d\omega_k\, e^{i(\omega_k-\omega)(t-\tau)}}_{\sim 2\pi\delta(t-\tau)}$$

Similarly to (257) one finds

$$\langle R_2(t) R_1(\tau)\rangle \ldots \tilde{a}^+(t) \ldots \tilde{a}(\tau) \ldots$$

(252)
$$\sim \sum_k g_k^2 e^{-i(\omega_k-\omega)(t-\tau)} (\bar{n}_k+1)$$
$$= (\gamma - i\Delta)(\bar{n}(\omega)+1)\,\delta(t-\tau)$$

This then gives

$$\dot{\rho}(t) = -\int_{t_0}^{t} d\tau \Big\{ \langle R_1(t) R_2(\tau)\rangle \left[a(t) a^+(\tau)\rho(\tau) - a^+(\tau)\rho(\tau)a(t)\right]$$
$$- \langle R_2(\tau) R_1(t)\rangle \left[a(t)\rho(\tau) a^+(\tau) = \rho(\tau) a^+(\tau) a(t)\right]$$
$$+ \langle R_2(t) R_1(\tau)\rangle \left[a^+(t) a(\tau)\rho(\tau) - a(\tau)\rho(\tau) a^+(t)\right]$$
$$- \langle R_1(\tau) R_2(t)\rangle \left[a^+(t)\rho(\tau) a(\tau) - \rho(\tau) a(\tau) a^+(t)\right] \Big\}$$

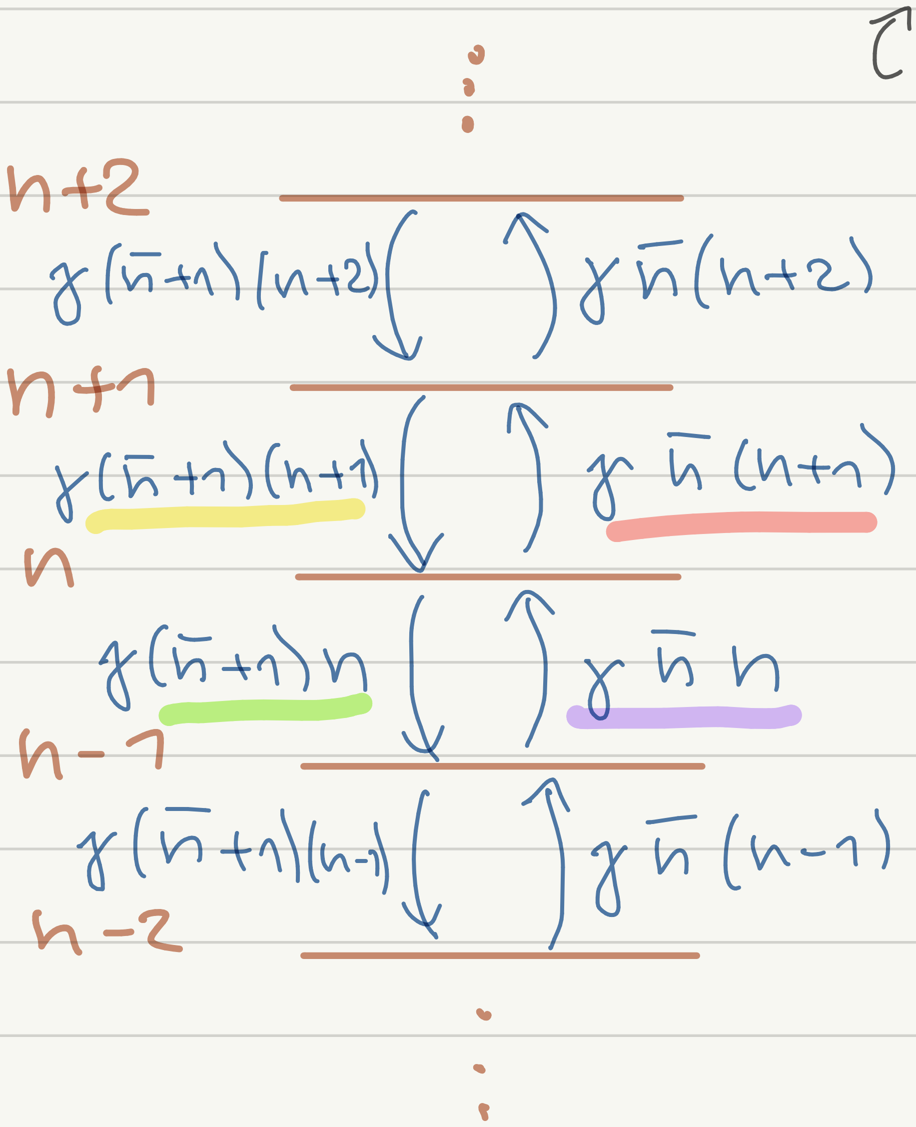
(253)
$$\boxed{\begin{aligned}\dot{\rho} &= -\frac{i}{2}\Delta\left[a^+a, \rho\right] \\ &+ \frac{\gamma}{2}(\bar{n}(\omega)+1)\left(2a\rho a^+ - a^+a\rho - \rho a^+a\right) \\ &+ \frac{\gamma}{2}\bar{n}(\omega)\left(2a^+\rho a - a a^+\rho - \rho a a^+\right)\end{aligned}}$$

interpretation $\quad p_n = \langle n|\rho|n\rangle \quad$ probability of excitation $= n$

$$\dot{p}_n = \gamma\left(\bar{n}(\omega)+1\right)\left((n+1)p_{n+1} - np_n\right)$$

(254) $$+\gamma\,\bar{n}(\omega)\left(np_{n-1} - (n+1)p_n\right)$$

probability currents



flow of probabilities

stationary state

$$\dot{p}_n^{ss} = 0 \qquad (255)$$

In stationary state the probability currents between p_n and p_{n+1} have to compensate each other

(255) $$\boxed{\gamma(\bar{n}+1)(n+1)p_{n+1}^{ss} = \gamma\bar{n}\,(n+1)\,p_n^{ss}}$$

this is called detailed balance

(256) $$\frac{p_{n+1}^{ss}}{p_n^{ss}} = \frac{\bar{n}}{\bar{n}+1}$$

(257) $$p_n^{ss} = \left(\frac{\bar{n}}{\bar{n}+1}\right)^n p_0^{ss} \quad \text{and} \quad \sum_{n=0}^{\infty} p_n^{ss} = 1$$

$$1 = \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1}\right)^n p_0^{ss} = \frac{1}{1-\frac{\bar{n}}{\bar{n}+1}} p_0^{ss}$$

$$\Rightarrow \quad p_0^{ss} = \frac{1}{\bar{n}+1}$$

(258) $$p_n^{ss} = \left(\frac{\bar{n}}{\bar{n}+1}\right)^n \frac{1}{\bar{n}+1} \qquad \bar{n} = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

this gives

(259) $$p_n^{ss} = e^{-\beta\hbar\omega n}\left(1-e^{-\beta\hbar\omega}\right)$$

Thus the stationary state of a harmonic oscillator coupled to a reservoir of harmonic oscillators at temperature $\beta = 1/k_B T$ is again a thermal state with the same temperature!

$\Rightarrow$ thermalization

(B) Two-level system coupled to reservoir of harmonic oscillators at $T \neq 0$

(260) $$H_S = \frac{1}{2}\hbar\omega\sigma_z \qquad H_R = \sum_k \hbar\omega_k\left(b_k^+ b_k + \frac{1}{2}\right)$$

(261) $$H_{SR} = \hbar\sum g_k\left(\sigma^- b_k^+ + \sigma^+ b_k\right)$$

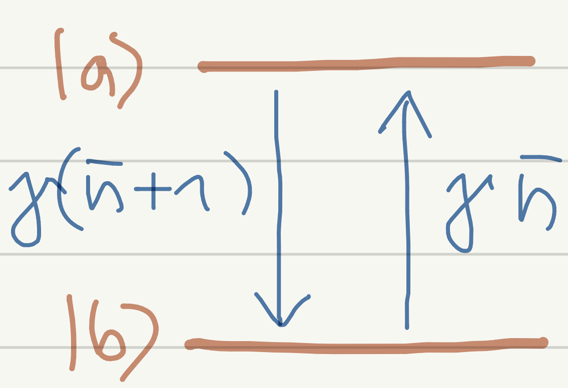
fully analogous to (A) with $\hat{a} \to \sigma^-$ and $\hat{a}^+ \to \sigma^+$

$\circ\circ\circ>$ without term in Hamiltonian

$$\boxed{\begin{aligned} \dot{\rho}\Big|_{diss} &= \frac{\gamma}{2}\left(\bar{n}(\omega)+1\right)\left(2\sigma^-\rho\,\sigma^+ - \sigma^+\sigma^-\rho - \rho\,\sigma^+\sigma^-\right) \\ &+ \frac{\gamma}{2}\,\bar{n}(\omega)\left(2\sigma^+\rho\,\sigma^- - \sigma^-\sigma^+\rho - \rho\,\sigma^-\sigma^+\right) \end{aligned}} \tag{262}$$

notation $|a\rangle \equiv |\uparrow\rangle \qquad |b\rangle \equiv |\downarrow\rangle$

$$\boxed{\begin{aligned} \dot{\rho}_{aa} &= -\gamma\left(\bar{n}+1\right)\rho_{aa} + \gamma\,\bar{n}\,\rho_{bb} \\ \dot{\rho}_{bb} &= \gamma\left(\bar{n}+1\right)\rho_{aa} - \gamma\,\bar{n}\,\rho_{bb} \\ \dot{\rho}_{ab} &= -\frac{1}{2}\gamma\left(2\bar{n}+1\right)\rho_{ab} \end{aligned}} \tag{263}$$



$\rho_{bb} + \rho_{aa} = 1 \qquad \rho_{bb} = 1 - \rho_{aa}$

stationary state

$$\rho_{aa}^{ss}\left(2\bar{n}+1\right) = \bar{n} \tag{264}$$

$$\underline{\underline{\rho_{aa}^{ss} = \frac{\bar{n}}{2\bar{n}+1}}} \xrightarrow[\bar{n}\to\infty]{} \frac{1}{2}$$

- decay rate of off-diagonal ρ_{ab}

$\frac{1}{2}$ * sum of decay rates of $|a\rangle$ and $|b\rangle$

- generic behavior

$$\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ab} & \rho_{bb} \end{pmatrix} \longrightarrow \begin{pmatrix} \rho_{aa} & 0 \\ 0 & \rho_{bb} \end{pmatrix}$$

off-diagonal matrix element decay
"decoherence"

there can be an additional phase decay

(265) $$\hat{L}_{deph} = \sqrt{\gamma_d}\, \sigma_z$$

(266) $$\boxed{\dot{\rho}\Big|_{deph} = \gamma_d \left(\sigma_z \rho \sigma_z - \rho \right)}$$

here $\dot{\rho}_{aa} = \dot{\rho}_{bb} = 0$

(267) $$\dot{\rho}_{ab}\Big|_{deph} = -\gamma_d \rho_{ab}$$

so decay rate of ρ_{ab}

$\frac{1}{2}$ * sum of decay from states $+ \gamma_d$