(65) $i\hbar \frac{d}{dt} P|\psi(t)\rangle = \underbrace{\left[PHP - PHQ\,(QHQ)^{-1} QHP \right]}_{H_{eff}} P|\psi\rangle$

This gives an effective Hamiltonian for the slow low-energy dynamics

(66) $$\boxed{H_{eff} = PHP - PH_1 Q \frac{1}{QHQ} QH_1 P}$$

effective Hamiltonian in adiabatic approxim.

where we used that P and Q are orthogonal projectors to subspaces spanned by H_0

II. Photons

Until now we have considered charged quantum particles in external, i.e. prescribed electromagnetic fields. This is not completely satisfactorily

consider e.g. Lorentz force

classical physics $\vec{E} = -\frac{\vec{F}}{q}$

quantum physics $? = -\frac{\hat{\vec{F}}}{q}$ operator?!

This motivates us to ask for a quantum theory of the elm. field.

A full quantum theory of fields $\Longrightarrow$ QM $\underline{\text{III}}$

<u>Here!</u> simplified but useful consideration

We will see that we can consider the electromagnetic field as a system of harmonic oscillators which we then replace by quantum harmonic oscillators.

II.1 Quantization of radiation field

Maxwell equations (SI units)

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \end{aligned}} \quad (67) \qquad \text{homogeneous MWE}$$

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \\ &\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} = \mu_0\vec{j} \end{aligned}} \quad (68) \qquad \text{inhomogeneous MWE}$$

$$c^2 = \frac{1}{\mu_0\varepsilon_0}$$

only consistent if

$$\boxed{\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0} \quad (69) \qquad \text{continuity equation}$$

since $\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} = \frac{\partial}{\partial t}\rho/\varepsilon_0$ in (68)

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{=0} - \mu_0\vec{\nabla}\vec{j} = \frac{\partial}{\partial t}\rho/\varepsilon_0$$

(A) general solution of homogeneous MWE

(70) $$\boxed{\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}}$$

$\vec{A}$: vector potential ; ϕ : scalar potential

$\Rightarrow$ eqs. (67) automatically solved

- consider charge- and current-free space

$\vec{j} = 0$; $\rho = 0$

(71) $$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = 0$$

plugging in eqs. (70) leads to coupled equations between $\vec{A}$ and ϕ. These can be decoupled by making use of gauge freedom

(72) $$\boxed{\begin{aligned}\vec{A} &\to \vec{A}' = \vec{A} + \vec{\nabla}\lambda \\ \phi &\to \phi' = \phi + \frac{\partial\lambda}{\partial t}\end{aligned}}$$ λ arbitrary

which leaves physical fields $\vec{E}$ and $\vec{B}$ unchanged.

radiation (Coulomb) gauge (note $\vec{j}=0, \varrho=0$)

(73) $\vec{\nabla}\cdot\vec{A}=0 \qquad \phi=0$

then one finds a wave equation for $\vec{A}$

(74) $$\boxed{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\vec{A}(\vec{r},t)=0}$$

The solutions of this equation are plane waves which are transversal

(75) $$\vec{A}(\vec{r},t)\sim\vec{e}(\vec{k})\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

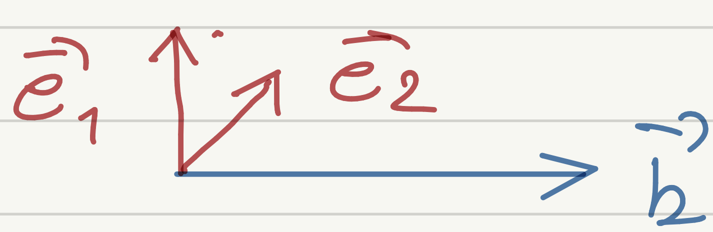
with

(76) $$\boxed{\omega=|\vec{k}|c} \qquad \boxed{\vec{e}(\vec{k})\cdot\vec{k}=0}$$

transversality $\vec{\nabla}\cdot\vec{A}=0$

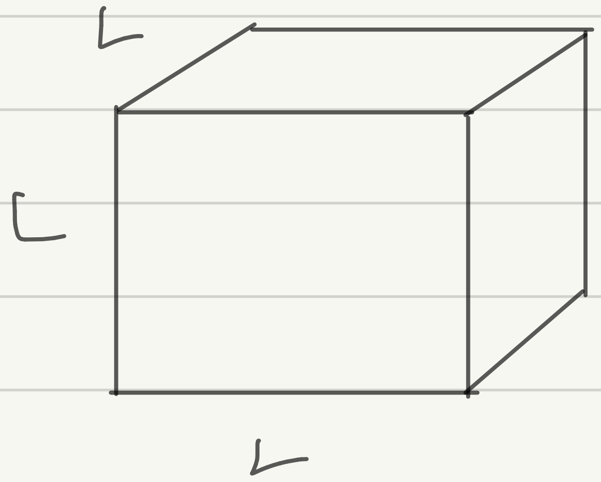
For a given wave-vector $\vec{k}$ there are only two independent polarizations

(77) $$\vec{e}_1(\vec{k})\cdot\vec{e}_2(\vec{k})=0 \qquad \vec{e}_i(\vec{k})\cdot\vec{k}=0$$



e.g. linear polarizations

To treat fields in an infinitely extended space has some mathematical problems. We therefore consider a finite volume $V = L^3$ with periodic boundary conditions and let $L \to \infty$ at the end.



$\Rightarrow$ discrete values of $\vec{k}$

$$\vec{k}_n = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$n_i \in \mathbb{Z} \,/\, (0,0,0)$$

General Fourier-representation of $\vec{A}(\vec{r},t)$

$$(78) \quad \vec{A}(\vec{r},t) = \frac{1}{\sqrt{2\varepsilon_0 L^3}} \sum_{\vec{k}_n} \sum_{\lambda=1,2} \vec{e}_\lambda(\vec{k})\, q_\lambda(\vec{k},t)\, e^{i\vec{k}\cdot\vec{r}}$$

- Since $\vec{A}^* = \vec{A} \Rightarrow$

$$q_\lambda(-\vec{k},t) \overset{!}{=} q_\lambda^*(\vec{k},t) \qquad \vec{e}_\lambda(-\vec{k}) \overset{!}{=} \vec{e}_\lambda^*(\vec{k})$$

- In order to exclude $\vec{k} = (0,0,0)$ we choose

$$q_\lambda(\vec{0},t) \equiv 0$$

- To fulfill the wave equation (74)

$$(79) \quad \ddot{q}_\lambda(\vec{k},t) + \omega_{\vec{k}}^2\, q_\lambda(\vec{k},t) = 0$$

This is the equation for the position of an harmonic oscillator!

(B) Energy and Hamilton function

(80) $$E = \int d^3r \left(\frac{\varepsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2 \right)$$

$$= \int d^3r \left(\frac{\varepsilon_0}{2} |\dot{\vec{A}}|^2 + \frac{1}{2\mu_0} |\vec{\nabla}\times\vec{A}|^2 \right)$$

using

$$(\vec{\nabla}\times\vec{A})^2 = \vec{\nabla}\cdot(\vec{A}\times(\vec{\nabla}\times\vec{A})) + \vec{A}\cdot\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{A})}_{\substack{=0 \\ \text{Coulomb gauge}}} - \vec{A}\Delta\vec{A}$$

and $$\int_{V_\infty} d^3r\, \vec{\nabla}\cdot(\vec{A}\times(\vec{\nabla}\times\vec{A})) = 0$$

using Gauß

we find

$$E = \frac{\varepsilon_0}{2}\int d^3r \left(\dot{\vec{A}}^2 - c^2 \vec{A}\Delta\vec{A} \right)$$

Substituting the Fourier-decomposition (78) finally yields

(81) $$E = \frac{1}{4}\sum_{\vec{k}\lambda} \dot{q}_\lambda^*(\vec{k},t)\dot{q}_\lambda(\vec{k},t) + \omega_k^2\, q_\lambda^*(\vec{k},t)\, q_\lambda(\vec{k},t)$$

decomposing $q_\lambda(\vec{k},t)$ into real and imaginary parts

$$q_\lambda = q_{1\lambda} + i\, q_{2\lambda} \qquad q_{i\lambda} \in \mathbb{R}$$

with condition on $q_\lambda^*(-\vec{k}) = q_\lambda(\vec{k})$ one has

(82) $$q_{1\lambda}(-\vec{k}) = q_{1\lambda}(\vec{k})$$ symmetric in $\vec{k}$

$\Rightarrow$ summation only over positive $\vec{k}$ (half space)

(83) $$\boxed{H = E = \frac{1}{2} \sum_{\vec{k},\lambda}' \left[\dot{q}_{1\lambda}^2 + \omega_k^2 q_{1\lambda}^2 + \dot{q}_{2\lambda}^2 + \omega_k^2 q_{2\lambda}^2 \right]}$$

free elm. field = sum of harmonic oscillators

(C) Quantization of elm. field

classical harmonic oscillators

$\downarrow$

quantum harmonic oscillators

canonical momenta (mass = 1)

(84) $$p_{i\lambda}(\vec{k}) = \dot{q}_{i\lambda}(\vec{k})$$

$$q_{i\lambda} \to \hat{q}_{i\lambda} \qquad p_{i\lambda} \to \hat{p}_{i\lambda}$$

(85) $$\boxed{[\hat{q}_{i\lambda}, \hat{p}_{j\lambda'}] = i\hbar \delta_{ij} \delta_{\lambda\lambda'}}$$ (same $\vec{k}$)

(86) $$H = \frac{1}{2} \sum_{\vec{k},\lambda}' \left[\hat{p}_{1\lambda}^2 + \omega_k^2 \hat{q}_{1\lambda}^2 + \hat{p}_{2\lambda}^2 + \omega_k^2 \hat{q}_{2\lambda}^2 \right]$$

Quantum mechanics of harmonic oscillators uses
ladder operators

(87)
$$\hat{a}_{j\lambda}(\vec{k}) \equiv \sqrt{\frac{\omega_k}{2\hbar}}\left[\hat{q}_{j\lambda}(\vec{k}) + \frac{i}{\omega_{\vec{k}}}\hat{p}_{j\lambda}(\vec{k})\right]$$
$$\hat{a}^{+}_{j\lambda}(\vec{k}) = \sqrt{\frac{\omega_k}{2\hbar}}\left[\hat{q}_{j\lambda}(\vec{k}) - \frac{i}{\omega_{\vec{k}}}\hat{p}_{j\lambda}(\vec{k})\right]$$

(88)
$$\left[\hat{a}_{i\lambda}(\vec{k}), \hat{a}_{j\lambda'}(\vec{k}')\right] = \delta_{ij}\delta_{\lambda\lambda'}\delta_{\vec{k}\vec{k}'}$$

so far $\vec{k}, \vec{k}'$ only in half space $\Rightarrow$ simplification

(89)
$$\hat{a}_{\lambda}(\vec{k}) \equiv \frac{1}{\sqrt{2}}\left[\hat{a}_{1\lambda}(\vec{k}) + i\,\hat{a}_{2\lambda}(\vec{k})\right]$$
$$\hat{a}_{\lambda}(-\vec{k}) \equiv \frac{1}{\sqrt{2}}\left[\hat{a}_{1\lambda}(\vec{k}) - i\,\hat{a}_{2\lambda}(\vec{k})\right]$$

now all $\vec{k}, \vec{k}'$ and

(90)
$$\boxed{\left[\hat{a}_{\lambda}(\vec{k}), \hat{a}^{+}_{\lambda'}(\vec{k}')\right] = \delta_{\lambda\lambda'}\delta_{\vec{k}\vec{k}'}}$$

with this we eventually arrive at (Heisenberg pict.)

(91)
$$\boxed{\hat{\vec{A}}(\vec{r},t) = \sqrt{\frac{\hbar}{4\pi\varepsilon_0 L^3}}\sum_{\vec{k},\lambda}\frac{1}{\sqrt{\omega_k}}\,\vec{e}_{\lambda}(\vec{k})\cdot\left[\hat{a}_{\lambda}(\vec{k},t)\,e^{i\vec{k}\cdot\vec{r}} + \hat{a}^{+}_{\lambda}(\vec{k},t)\,e^{-i\vec{k}\cdot\vec{r}}\right]}$$

Hamiltonoperator

(92) $$\boxed{H = \sum_{\vec{k},\lambda} \hbar\omega_{\vec{k}} \left(\hat{a}^\dagger_\lambda(\vec{k})\, \hat{a}_\lambda(\vec{k}) + \frac{1}{2} \right)}$$

divergent zero-point energy $\Rightarrow$ ignore at this point!

interpretation

(93) $$\boxed{\hat{n}_\lambda(\vec{k}) = \hat{a}^\dagger_\lambda(\vec{k})\, \hat{a}_\lambda(\vec{k})}$$

number operator in mode $(\vec{k},\lambda)$

$\Longrightarrow$ photons

$\hat{a}^\dagger_\lambda(\vec{k})$ generates a photon with momentum $\vec{k}$ and polarization λ

$\hat{a}_\lambda(\vec{k})$ annihilates a photon with momentum $\vec{k}$ and polarization λ

$\hbar\omega_{\vec{k}}$: energy of a photon with momentum $\vec{k}$

(D) Momentum and angular momentum of photons

classical EDyn

(94) $$\vec{P} = \varepsilon_0 \int d^3\vec{r}\; \vec{E} \times \vec{B}$$

Substituting the mode decomposition of $\vec{A}$ yields after some steps

(85) $$\boxed{\hat{\vec{P}} = \sum_{\vec{k}\lambda} \hbar \vec{k}\, \hat{a}^{+}_{\lambda}(\vec{k})\, \hat{a}_{\lambda}(\vec{k})}$$

- interpretation! each photon carries a momentum of $\hbar\vec{k}$

classical Edyn: angular momentum with respect to $\vec{r}_0$ (as in mechanics $\vec{L} = (\vec{r}-\vec{r}_0)\times\vec{p}$)

(96) $$\vec{L}_G = \int d^3r\, (\vec{r}-\vec{r}_0) \times (\vec{E}\times\vec{B})$$

one finds that $\vec{L}$ can be written as the sum of two terms

(97) $$\boxed{\vec{L}_G = \vec{L} + \vec{S}}$$

depends on choice of $\vec{r}_0$

orbital angular momentum

does not depend on $\vec{r}_0$

intrinsic angular momentum (spin)

$S = 1$ per photon

more in QFTh (QM II)

II.2 Spontaneous Emission

One of the most important consequences of the quantization of the radiation field is spontaneous emission conjectured to exist by Einstein.

Hamilton operator of bound electron in quantized elm. field in Coulomb gauge

(98) $$H_e = \frac{1}{2m_e}(\vec{p} - e\vec{A})^2 + V(\vec{r}) = \underbrace{\frac{\vec{p}^2}{2m_e}}_{} - \underbrace{\frac{e}{m_e}\vec{p}\cdot\vec{A}}_{} + \cancel{\frac{e^2}{2m}\vec{A}^2} + V$$

as in I.3

$$= H_0 + H_1$$

Now we need to add the free Hamiltonian of the radiation field

(99) $$H = H_0 + H_\gamma \qquad H_\gamma = \sum_{\vec{k},\lambda} \omega_{\vec{k}}\, \vec{a}^{\dagger}_{\lambda}(\vec{k})\, \vec{a}_{\lambda}(\vec{k})$$

- time-dependent perturbation theory

photon vacuum

initial state $\quad |i\rangle = |n; 0\rangle$

final state $\quad |f\rangle = |m; 1_{\vec{k}\lambda}\rangle$ $\quad$ + summation over $\vec{k}, \lambda$

$$\Delta E = E_f - E_i = E_m - E_n + \hbar\omega_{\vec{k}}$$

- Fermi's golden rule $\quad \left(\frac{1}{L^3}\sum_{k} \rightarrow \frac{1}{(2\pi)^3}\int d^3k\right)$

(100) $$W_{i\to f} = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \frac{2\pi}{\hbar}\, \delta(E_m - E_n + \hbar\omega_{\vec{k}})\, |\langle f|H_1|i\rangle|^2$$

matrix element

$$\langle f|H_1|i\rangle = -\frac{e}{m}\langle m|\langle 1_{\vec{k}\lambda}|\hat{\vec{A}}(\vec{r})|0\rangle\cdot\hat{\vec{p}}|n\rangle \tag{101}$$

now

$$|1_{\vec{k}\lambda}\rangle = \hat{a}^{+}_{\lambda}(\vec{k})|0\rangle$$

$$\langle 1_{\vec{k}\lambda}|\hat{\vec{A}}(\vec{r})|0\rangle = \langle 0|\hat{a}_{\lambda}(\vec{k})\hat{\vec{A}}(\vec{r})|0\rangle$$

$$= \sqrt{\frac{\hbar}{4\pi\varepsilon_0}}\int\frac{d^3\vec{k}'}{(2\pi)^3}\frac{1}{\sqrt{\omega_{k'}}}\sum_{\lambda'}\vec{e}_{\lambda'}(\vec{k})e^{-i\vec{k}'\vec{r}} \tag{102}$$

$$\cdot\underbrace{\langle 0|\hat{a}_{\lambda}(\vec{k})\hat{a}^{+}_{\lambda'}(\vec{k}')|0\rangle}_{=\delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}')(2\pi)^3}$$

due to continuum limit

gives contribution even in vacuum field!

All further steps as for stimulated emission in Sec. II

$$\boxed{W_{n\to m} = \frac{4}{3}\frac{\omega^3}{\hbar c^3(4\pi\varepsilon_0)}|\langle m|\vec{d}|n\rangle|^2} \tag{103}$$

Einstein A-coefficient of spontaneous emission