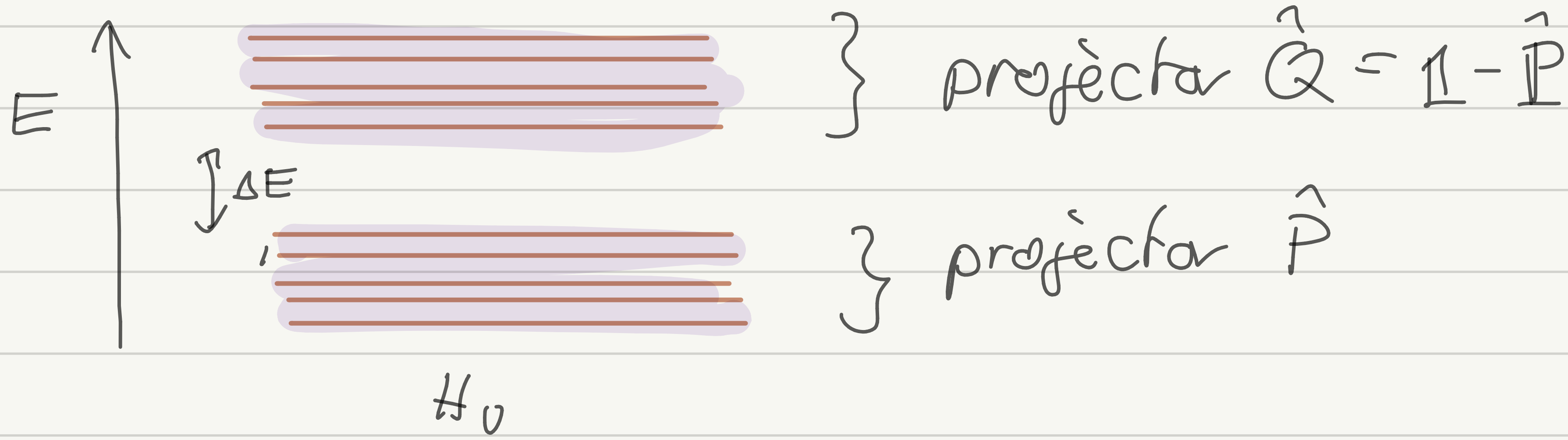
I.4 Adiabatic Elimination

Let us consider again a hamiltonian

$$H = H_0 + H_1 \tag{59}$$

where the spectrum of H_0 separates in clearly distinct energy regions and H_1 is a perturbation



Consider projectors onto subspaces of H_0

$$\hat{\mathbb{P}} \text{ and } \hat{Q} \qquad \hat{\mathbb{P}}^2 = \hat{\mathbb{P}} \quad \hat{Q}^2 = \hat{Q}$$

Assume further that

$$|\hat{\mathbb{P}} H \hat{Q}| = |\hat{\mathbb{P}} H_1 \hat{Q}| \tag{60}$$

is small compared to

$$\min\left(|QHQ| - |PHP|\right) \sim \Delta E$$

If the initial state of a quantum evolution is a low energy state such that

$$(61) \qquad |\Psi_0\rangle = \hat{\mathbb{P}}\,|\Psi_0\rangle$$

then the time evolution remains approximately in the subspace of low-energy states spanned by $\hat{\mathbb{P}}$ and we can derive an effective Hamiltonian

$$i\hbar \frac{d}{dt}\, \hat{\mathbb{P}}\,|\Psi(t)\rangle = \hat{\mathbb{P}} H |\Psi(t)\rangle$$

$$(62) \qquad = \left(\hat{\mathbb{P}} H \hat{\mathbb{P}} + \hat{\mathbb{P}} H \hat{Q}\right) |\Psi(t)\rangle$$

$$= PHP\, P|\Psi(t)\rangle + PHQ\, Q|\Psi(t)\rangle$$

and

$$i\hbar \frac{d}{dt} \hat{Q}|\Psi(t)\rangle = \hat{Q} H |\Psi(t)\rangle$$

$$(63) \qquad = \left(QHP + QHQ\right)|\Psi(t)\rangle$$

$$= QHP\, P|\Psi(t)\rangle + QHQ\, Q|\Psi(t)\rangle$$

since QHQ is a high energy, we can approximate.

$$(64) \qquad \left| QHQ\, Q|\Psi(t)\rangle \right| \gg \left| i\hbar \frac{d}{dt} Q|\Psi\rangle \right|$$

$\Rightarrow$

$$Q|\Psi(t)\rangle \simeq -\left(QHQ\right)^{-1} QHP\, P|\Psi(t)\rangle$$

substituting into (62) yields

$$(65)\quad i\hbar \frac{d}{dt} P|\psi(t)\rangle = \underbrace{\left[PHP - PHQ\,(QHQ)^{-1}\,QHP \right]}_{H_{eff}} P|\psi\rangle$$

This gives an effective Hamiltonian for the slow low-energy dynamics

$$(66)\quad \boxed{H_{eff} = PHP - PH_1 Q \frac{1}{QHQ} QH_1 P}$$

effective Hamiltonian in adiabatic approxim.

where we used that P and Q are orthogonal projectors to subspaces spanned by H_0

II. Photons

Until now we have considered charged quantum particles in external, i.e. prescribed electromagnetic fields. This is not completely satisfactorily

consider e.g. Lorentz force

classical physics $\quad \vec{E} \equiv -\frac{\vec{F}}{q}$

quantum physics $\quad ? \equiv -\frac{\hat{\vec{F}}}{q}\quad$ operator?!