Note: Please upload your assignment in the olat course under "Übungsaufgaben". Only the problems marked by a ☗ have to be submitted.

☗ **Problem 28.** *Phase Operator 1* (6 points)
The position representation of the angular momentum operator in the z-direction is given by

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}. \tag{1}$$

This suggests the introduction of a complementary "*phase operator*"

$$\hat{\phi} = \varphi. \tag{2}$$

(a) What is the commutation relation $\left[\hat{\phi}, \hat{L}_z\right]$ and what uncertainty relation does it imply?

(b) Consider an eigenstate $Y_{lm}(\vartheta, \varphi) \in \mathcal{L}^2(S^2, \mathrm{d}\Omega)$ of $\hat{L}_z$ and calculate $\langle \Delta\hat{\phi}^2 \rangle$. Show the inconsistency of introducing the phase operator (2) using this result.

Problem 29. *Phase Operator 2*
The inconsistency of the phase operator defined in Problem 28 arises because its application to states from $\mathcal{H} = \mathcal{L}^2(S^2, \mathrm{d}\Omega)$ leads to non-periodic functions and thus takes them out of $\mathcal{H}$. This can be resolved by introducing operators with the position representation

$$\widehat{\sin\phi} = \sin\varphi \qquad \widehat{\cos\phi} = \cos\varphi. \tag{3}$$

(a) Show that the following commutation relations hold:

$$\left[\widehat{\sin\phi}, \hat{L}_z\right] = i\hbar\, \widehat{\cos\phi} \tag{4}$$

$$\left[\widehat{\cos\phi}, \hat{L}_z\right] = -i\hbar\, \widehat{\sin\phi} \tag{5}$$

(b) What are the corresponding uncertainty relations?

Problem 30. *Time-dependent Hamiltonian: Exact Solution* (6 points)
Consider the driven harmonic oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2} m\omega^2 \hat{x}^2 - F\hat{x}\cos(\Omega t) \tag{6}$$

$$= \hbar\omega\left(\hat{a}^\dagger \hat{a} + \tfrac{1}{2}\right) - \hbar A\left(\hat{a}^\dagger + \hat{a}\right)\cos(\Omega t) \quad \text{for } A = \frac{F}{\sqrt{2m\hbar\Omega}}. \tag{7}$$

The energy eigenstates of the harmonic oscillator (with $F = 0$) are $|n\rangle$ for $n = 0, 1, 2, ...$ and

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{8}$$

Show that the state

$$|\psi_0(t)\rangle = e^{-i\theta(t)}\, e^{-\frac{1}{2}|\alpha(t)|^2} \sum_{n=0}^{\infty} \frac{\alpha(t)^n}{\sqrt{n!}}\,|n\rangle \tag{9}$$

is a solution of the time-dependent Schrödinger equation if $\theta(t)$ and $\alpha(t)$ satisfy the following conditions:

$$\dot{\alpha}(t) = -i\omega\alpha(t) + iA\cos(\Omega t) \tag{10}$$

$$\dot{\theta}(t) = \frac{\omega}{2} - \frac{A}{2}\left(\alpha(t) + \alpha(t)^*\right)\cos(\Omega t)\,. \tag{11}$$

Problem 31. *Runge-Lenz vector* (4 points)
Show that the Runge-Lenz vector

$$\hat{\vec{F}} = \frac{1}{2m}\left(\hat{\vec{p}} \times \hat{\vec{L}} - \hat{\vec{L}} \times \hat{\vec{p}}\right) - \frac{\alpha}{\hat{r}}\hat{\vec{r}} \tag{12}$$

commutes with the Hamiltonian $\hat{H} = \hat{\vec{p}}^2/2m - \alpha/r$ of the Coulomb potential.