Note: Please upload your assignment in the olat course under "Übungsaufgaben". Only the problems marked by a ☗ have to be submitted.

☗ **Problem 21.** *Angular momentum; ladder operators 1* (8 points)
In Cartesian coordinates holds:

$$\hat{\vec{L}}^2 = \frac{1}{2}\left(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right) + \hat{L}_z^2, \tag{1}$$

where $\hat{L}_\pm = \hat{L}_x \pm \mathrm{i}\hat{L}_y$ are the ladder operators of the angular momentum.

(a) Making use of the representation of the angular momentum operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$ in spherical coordinates, detrmine the representation of the ladder operators in the spherical coordinates.

(b) Derive with the help of (1) the representation of $\hat{\vec{L}}^2$ in spherical coordinates and show that in three dimensions the Laplace operator in spherical coordinates can be expressed through $\hat{\vec{L}}^2$ by:

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\hat{\vec{L}}^2}{\hbar^2 r^2} \tag{2}$$

☗ **Problem 22.** *Angular momentum; ladder operators 2* (6 points)
The matrices

$$\hat{L}_x \equiv \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \hat{L}_y \equiv \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \hat{L}_z = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

are the spatial components of the angular momentum for $l = 1$ in the basis of eigenstates $|m = -1\rangle, |m = 0\rangle, |m = 1\rangle$ of $\hat{L}_z$.

(a) Show that the eigenvalue of $\hat{\vec{L}}^2$ is $\hbar^2 l(l+1)$ with $l = 1$.

(b) Determine the eigenvectors of $\hat{L}_x, \hat{L}_y$ and $\hat{L}_z$ for the eigenvalue $m = 0$. Show that this set of vectors yields an orthogonal complete basis.

♠ **Problem 23.** *Spin 1* (6 points)

(a) Show that the spin matrices

$$\hat{s}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{s}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \hat{s}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

obey the commutation relations $[\hat{s}_i, \hat{s}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{s}_k$ and $[\hat{s}_i, \hat{\vec{s}}^{\,2}] = 0$. What is $\hat{\vec{s}}^{\,2}$ and determine the eigenvalues of the operator.

(b) The eigenstates (eigenspinors) of $\hat{s}_z$ are

$$\Psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Which linear combinations of $\Psi_\pm$ are eigenspinors of $\hat{s}_x$ and $\hat{s}_y$?

Problem 24. *Spin 2*

The general spin-components of a spin-1/2 system is $\hat{s}_{\vec{n}} := \frac{1}{2}\vec{n}\hat{\vec{\sigma}}$, where $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$ is the generalized unity vector. Determine the eigenvalues of $\hat{s}_{\vec{n}}$ and a complete set of orthonormal eigenstates. Determine the expectation value $\langle\hat{\sigma}_z\rangle$ in these eigenstates.