Note: Please upload your assignment in the olat course under "Übungsaufgaben". Only the problems marked by a ☗ have to be submitted.

☗ **Problem 18.** *Half harmonic oscillator* (6 points)
Consider a harmonic oscillator in one dimension with frequency ω and mass m. The corresponding energy eigenvalues and functions are E_n and $\phi_n(x)$. An infinite potential well is added at $x = 0$

$$V(x) = \begin{cases} \infty & \text{for } x \leq 0 \\ \frac{m\omega^2}{2}x^2 & \text{for } x > 0. \end{cases} \tag{1}$$

(a) Explain why the eigenfunction of the harmonic oscillator with the potential barrier, besides a normalization factor, is a subset of eigenfunctions ϕ_n of the harmonic oszillator. What are the corresponding eigenenergies?

(b) Let $\Psi_0(x)$ be an arbitrary wave function in the potential V(x) at time $t = 0$. Show that at time $t = T$ the wave function evolves into its negative, i.e.

$$\Psi(x,T) = -\Psi(x,0) = -\Psi_0(x). \tag{2}$$

T is the period of the oscillation $\omega T = 2\pi$. What happens at $t = 2T$?

☗ **Problem 19.** *Sum rule* (8 points)
Consider a particle in a one dimensional Potential $V(x)$ which has bound states $|E_n\rangle$ to the eigenvalues E_n of the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \tag{3}$$

$$\hat{H}\,|E_n\rangle = E_n\,|E_n\rangle\,. \tag{4}$$

(a) Show that

$$\left[\left[\hat{x},\hat{H}\right],\hat{x}\right] = \frac{\hbar^2}{m}. \tag{5}$$

(b) Use (a) and the identity operator $\mathbb{1} = \sum_k |E_k\rangle\langle E_k|$, prove the sum rule of the dipole matrix element $\hat{d} = e\hat{x}$

$$\sum_k \omega_{kn}|\langle E_n|\,\hat{d}\,|E_k\rangle|^2 = \sum_k \omega_{kn}\,\langle E_n|\,\hat{d}\,|E_k\rangle\,\langle E_k|\,\hat{d}\,|E_n\rangle = \frac{\hbar e^2}{2m}, \tag{6}$$

where $\hbar\omega_{kn} = E_k - E_n$.

(c) Check explicitly the sum rule for the one-dimensional harmonic oscillator.

⬆ **Problem 20.** *Virial theorem* (6 points)

Given the Hamilton operator in one dimension

$$\hat{H} = \hat{H}_{\text{kin}} + \hat{H}_{\text{pot}} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{7}$$

(a) Show that in position space the commutator of $\hat{H}$ and $\hat{x}\hat{p}$ reads:

$$\frac{1}{i\hbar}\left[\hat{H}, \hat{x}\hat{p}\right] = \hat{x}\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right) - \frac{1}{m}\hat{p}^2. \tag{8}$$

(b) Let φ_E be an eigenstate of $\hat{H}$ with Energy E. Show from equation (8) that

$$\left\langle \frac{\hat{p}^2}{2m} \right\rangle_E = \frac{1}{2}\left\langle \hat{x}\frac{\mathrm{d}}{\mathrm{d}x}V(x) \right\rangle_E, \tag{9}$$

where $\langle \bullet \rangle_E$ denotes the expectation value in the state φ_E.

(c) Show that in the special case

$$V(\lambda x) = \lambda^n V(x), \tag{10}$$

i.e. if $V(x)$ is a homogeneous function of order n, holds:

$$\left\langle \hat{H}_{\text{kin}} \right\rangle_E = \frac{n}{2}\left\langle \hat{H}_{\text{pot}} \right\rangle_E \qquad \text{(Virial theorem)}. \tag{11}$$

Hint: Calculate the first derivative (10) with respect to λ.