Note: Please upload your assignment in the olat course under "Übungsaufgaben". Only the problems marked by a ☗ have to be submitted.

☗ **Problem 15.** *Coupled oscillators; normal coordinates* (10 points)

The eigenvalues of a single harmonic oscillator were derived in the lecture. Now consider two coupled oscillators

$$\hat{H} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{x}_1^2 + \hat{x}_2^2\right) + \gamma m\omega^2 \hat{x}_1\hat{x}_2, \tag{1}$$

where $0 < \gamma < 1$ characterizes the coupling strength.

(a) Determine the eigenvalues for a vanishing coupling strength ($\gamma = 0$). What is the degree of degeneracy of the corresponding eigenfunctions, i.e. how many eigenfunctions for the same eigenvalue exist?

(b) Introduce new variables ξ and η and write $x_1 = \frac{1}{\sqrt{2}}(\xi + \eta)$ and $x_2 = \frac{1}{\sqrt{2}}(\xi - \eta)$. What is the Hamiltonian in the new coordinates and what are the corresponding momenta?

(c) Determine the eigenvalues for $\gamma \neq 0$. Show that the coupling term in general removes the degeneracy found in (a).

Problem 16. *Hermite Polynomials*

(a) Prove that $e^{2\lambda\xi-\lambda^2}$ is a generating function of the Hermite polynomials (eq. (215) in the script) , i.e.

$$e^{2\lambda\xi-\lambda^2} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(\xi). \tag{2}$$

Hint: $e^{2\lambda\xi-\lambda^2} = e^{\xi^2} e^{-(\lambda-\xi)^2}$

(b) Show the symmetry property $H_n(-\xi) = (-1)^n H_n(\xi)$ and that the Hermite polynomials satisfy

$$H_n'(\xi) = 2nH_{n-1}(\xi), \tag{3}$$

$$2\xi H_n(\xi) = 2nH_{n-1}(\xi) + H_{n+1}(\xi). \tag{4}$$

Hint: Differentiate eq. (2) with respect to ξ and λ and compare the coefficients of λ^n.

Problem 17. *Coherent states* (8 points)

(a) Show that coherent states are normalized, i.e. $\langle\alpha|\alpha\rangle = 1$, but are not orthogonal. Determine $\langle\alpha|\beta\rangle$.

(b) Use the Baker-Hausdorff theorem to show that coherent states satisfy:

$$|\alpha\rangle = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)|0\rangle\,, \tag{5}$$

where $|0\rangle$ is the eigenstate of $\hat{n} = \hat{a}^\dagger\hat{a}$ for the eigenvalue 0.

(c) Show that the coherent states are overcomplete

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\ |\alpha\rangle\langle\alpha| = \mathbb{1}, \tag{6}$$

where $\int \mathrm{d}^2\alpha = \int \mathrm{d}u \int \mathrm{d}v = \int_0^\infty \mathrm{d}r\ r \int_0^{2\pi} \mathrm{d}\varphi$ with $\alpha = u + iv = re^{i\varphi}$.

Hint: Use $\int_0^\infty \mathrm{d}x\ x^n e^{-x} = n!$.