Note: Please upload your assignment in the olat course under "Übungsaufgaben". Only the problems marked by a ☗ have to be submitted.

☗ **Problem 9.** *Free fall (8 points)*
The one dimensional Schrödinger equation in the presence of a constant force $F = -mg$ (free fall) can be written as:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + mgx\right]\psi(x,t). \tag{1}$$

(a) Derive from the above equation a partial differential equation of first order for the wave function $\tilde{\psi}(k,t)$ in k-space.

(b) Show that the partial differential equation can be written as an ordinary differential equation, when substituting $k = k_0 + \frac{mg}{\hbar}t$

$$\frac{\partial}{\partial t}\tilde{\psi}(k_0 + \frac{mg}{\hbar}t, t) = -i\frac{\hbar}{2m}(k_0 + \frac{mg}{\hbar}t)^2\tilde{\psi}(k_0 + \frac{mg}{\hbar}t, t). \tag{2}$$

(c) Determine $\tilde{\psi}(k,t)$ and show that the average momentum $\langle\hat{p}\rangle = \hbar\langle k\rangle$ of a particle is a linear function in time t.

Problem 10. *Infinite square well*
Given be a one-dimensional square well with infinite walls, i.e.,

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq L \\ +\infty & \text{else.} \end{cases} \tag{3}$$

The stationary wave function $\phi_n(x)$ and the associate energy eigenvalues E_n were calculated in the lecture.

(a) Show that the general time depending solution of the Schrödinger equation with the initial condition $\psi(x,0) = \psi_0(x)$ can be written in the form

$$\psi(x,t) = \sum_n \alpha_n\phi_n(x)\mathrm{e}^{-\frac{i}{\hbar}E_n t}. \tag{4}$$

How are the α_n determined?

(b) At time $t = 0$ let the wave function be $\psi(x,t=0) = \psi_0(x)$. Let the period of the fundamental oscillation be T. Show that after a time $t = T/2$ the function evolves into its reflection with respect to $x = L/2$, i.e. that $\psi(x,T/2) = -\psi(L-x,0) = -\psi_0(L-x)$. What happens after multiples of T?

⬆ **Problem 11.** *Free fall II (8 points)*
Let us reconsider Problem 9.

(a) Show that the stationary wave function of energy E is given by

$$\phi_E(x) = \mathcal{N}\mathrm{Ai}\left(\frac{x}{l_0} - \frac{E}{\epsilon_0}\right). \tag{5}$$

Here $\mathcal{N}$ is a normalization constant, $l_0 = (\hbar^2/2m^2g)^{1/3}$ and $\epsilon_0 = (\hbar^2 mg^2/2)^{1/3}$ are the standard length and energy, respectively.

$$\mathrm{Ai}(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \exp\left(i(yx + \frac{1}{3}y^3)\right) \tag{6}$$

is the Airy function. First show that this function fulfills the differential equation

$$\mathrm{Ai}''(x) - x\mathrm{Ai}(x) = 0. \tag{7}$$

Its lowest zeros are $-2.3381\ldots$, $-4.0879\ldots$, $-5.52055\ldots$.

(b) Now the particle falls onto a hard surface at $x = 0$, meaning the potential be

$$V(x) = \begin{cases} mgx & \text{for } x > 0 \\ +\infty & \text{for } x \leq 0. \end{cases} \tag{8}$$

What has to hold for $\phi(x)$ when $x \leq 0$? What follows for the allowed energy values, if you use the above ansatz for the wave function in the case of $x > 0$?