Note: Please upload your assignment in the olat course under "Übungsaufgaben". Only the problems marked by a ☗ have to be submitted.

☗ **Problem 5.** *Operator derivatives (8 points)*
The derivative of a continuous operator $A(\lambda), \lambda \in \mathbb{R}$, is defined as

$$\frac{\mathrm{d}A(\lambda)}{\mathrm{d}\lambda} := \lim_{\epsilon \to 0} \frac{A(\lambda + \epsilon) - A(\lambda)}{\epsilon} \tag{1}$$

(provided the limit exist). Show:

(a)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(AB) = \frac{\mathrm{d}A}{\mathrm{d}\lambda}B + A\frac{\mathrm{d}B}{\mathrm{d}\lambda}; \tag{2}$$

(b) *

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}A^{-1} = -A^{-1}\frac{\mathrm{d}A}{\mathrm{d}\lambda}A^{-1}; \tag{3}$$

(c)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{e}^{\lambda C} = C\mathrm{e}^{\lambda C}, \quad \text{if} \quad C \neq C(\lambda); \tag{4}$$

(d)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}A^n = \sum_{l=1}^{n} A^{l-1}\frac{\mathrm{d}A}{\mathrm{d}\lambda}A^{n-l}, \quad n \in \mathbb{N}. \tag{5}$$

Determine especially for $A \neq A(\lambda), B \neq B(\lambda)$ the derivatives

$$(\mathbf{e}) \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathrm{e}^{\lambda B}A\mathrm{e}^{-\lambda B}) \quad \text{and} \quad (\mathbf{f}) \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathrm{e}^{\lambda A}\mathrm{e}^{\lambda B}). \tag{6}$$

*(Instruction to (b) : Use $A^{-1}A = \mathbb{1}$ and the product rule from (a).)

☗ **Problem 6.** *Matrix exponential (6 points)*
Given a selfadjoint matrix

$$\hat{A} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{7}$$

Determine the matrix exponential

$$\hat{T}(\alpha) := \mathrm{e}^{i\alpha\hat{A}}, \quad \alpha \in \mathbb{R}. \tag{8}$$

(a) Use the matrix exponential series (Taylor series).

(b) Use the spectral decomposition.

Problem 7. *Uncertainty principle (6 points)*

(a) Let $[\hat{B}, \hat{C}] = \hat{A}$ and $[\hat{A}, \hat{C}] = \hat{B}$ be two operators. Show that

$$\Delta(\hat{A}\hat{B})\Delta\hat{C} \geq \frac{1}{2}\langle \hat{A}^2 + \hat{B}^2 \rangle. \tag{9}$$

(b) In one spatial dimension holds:

$$\langle \Delta \hat{r}_\alpha^2 \rangle \langle \Delta \hat{p}_\alpha^2 \rangle \geq \frac{\hbar^2}{4}, \tag{10}$$

where $\alpha = x, y, z$. Show:

$$\langle \Delta \hat{\vec{r}}^{\,2} \rangle \langle \Delta \hat{\vec{p}}^{\,2} \rangle \geq \frac{9\hbar^2}{4}. \tag{11}$$

Use the simplifying assumption that $\langle \hat{r}_\alpha \rangle = \langle \hat{p}_\alpha \rangle = 0$ and use the inequality.

$$\frac{a}{b} + \frac{b}{a} \geq 2 \qquad \text{für } a, b > 0. \tag{12}$$

Problem 8. *Commutators*

(a) Let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be different operators. Show:

(i) $[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$,

(ii) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$.

(b) Let $\hat{A}$, $\hat{B}$ be hermitian operators. Show that $\hat{A}\hat{B}$ is hermitian, if $[\hat{A}, \hat{B}] = 0$.

(c) Let $\hat{A}$, $\hat{B}$ be two commuting operators and let $|a\rangle$ be the eigenstate of $\hat{A}$ with eigenvalue a. Show that $\hat{B}\,|a\rangle$ is also an eigenstate of $\hat{A}$. Find the corresponding eigenvalue to that state.