"*Anyone who is not shocked by quantum theory has not understood it.*" – Niels Bohr

Problem 1. *Operators*
The following operators are given:

$$
\begin{aligned}
&(i)\quad \hat{D} = \frac{\partial}{\partial x} \quad \text{with} \quad \hat{D}\psi(x) = \frac{\partial \psi(x)}{\partial x} \\
&(ii)\quad \hat{\mathbb{1}} \quad \text{with} \quad \hat{\mathbb{1}}\psi(x) = \psi(x) \\
&(iii)\quad \hat{F} \quad \text{with} \quad \hat{F}\psi(x) = f(x)\psi(x) \\
&(iv)\quad \hat{P} \quad \text{with} \quad \hat{P}\psi(x) = \psi(x)^3 + 3\psi(x)^2 - 4 \\
&(v)\quad \hat{Q} \quad \text{with} \quad \hat{Q}\psi(x) = \int_0^1 \mathrm{d}x \psi(x).
\end{aligned}
$$

(a) Which of these operators are linear?
(b) $\hat{A}^{-1}$ with $\hat{A} = \hat{D}, \hat{\mathbb{1}}, \hat{F}$ is the inverse of A iff the inverse of A exist. Construct the inverse of A.
(c) Show that $i\hat{D}$ is hermitian.

Problem 2. *Adjoint Operators*
Show:

(a) $\left(\hat{A}\hat{B}\right)^\dagger = \hat{B}^\dagger \hat{A}^\dagger$
(b) $(\hat{A}^\dagger)^\dagger = \hat{A}$
(c) If an operator $\hat{B}$ has the eigenvalue b with $b \neq b^*$, then $\hat{B} \neq \hat{B}^\dagger$.

The *commutator* of two operators $\hat{A}, \hat{B}$ is defined as

$$\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{1}$$

(d) Show: If $\hat{A} = \hat{A}^\dagger$ and $\hat{B} = \hat{B}^\dagger$ then

$$\left[\hat{A}, \hat{B}\right]^\dagger = -\left[\hat{A}, \hat{B}\right] \tag{2}$$

Problem 3. *Complex Conjugation*
Consider an operator $\hat{C}$ with the following property

$$\hat{C}\psi(x) = \psi(x)^* \tag{3}$$

(a) Is $\hat{C}$ hermitian?
(b) What are the eigenfunctions of $\hat{C}$?
(c) What are the eigenvalues of $\hat{C}$?

Problem 4. *Scalar product and expectation values*

(a) Show that the scalar product of two vectors $|f\rangle$ and $|g\rangle$ can be written as an integral in position space or momentum space respectively

$$\int \mathrm{d}x\ f^*(x)g(x) \qquad \text{or} \qquad \int \mathrm{d}k\ \tilde{f}^*(k)\tilde{g}(k). \tag{4}$$

(b) Show that since $\langle f|f\rangle = 1$ it follows

$$\int \mathrm{d}x\ |f(x)|^2 = 1 \qquad \text{or} \qquad \int \mathrm{d}k\ |\tilde{f}(k)|^2 = 1. \tag{5}$$

(c) Consider the Gaussian wave function

$$\Psi(x) = \mathcal{N}\mathrm{e}^{-x^2/d^2}. \tag{6}$$

Compute the normalization factor $\mathcal{N}$ and the expectation values

$$(i) \quad \langle \hat{x}\rangle = \langle\Psi|\,\hat{x}\,|\Psi\rangle \tag{7}$$
$$(ii) \quad \langle \hat{x}^2\rangle = \langle\Psi|\,\hat{x}^2\,|\Psi\rangle \tag{8}$$
$$(iii) \quad \langle \hat{p}\rangle = \langle\Psi|\,\hat{p}\,|\Psi\rangle \tag{9}$$
$$(iv) \quad \langle \hat{p}^2\rangle = \langle\Psi|\,\hat{p}^2\,|\Psi\rangle\,, \tag{10}$$

where $\hat{p} = -i\hbar\partial_x$.