Note: Please upload your assignment in the olat course under "Übungsaufgaben". Only the problems marked by a ☗ have to be submitted.

☗ **Problem 32.** *Kinetic momentum* (4 points)
Different from the canonical momentum $\hat{\vec{p}}$ the kinetic momentum $m\hat{\vec{v}} = \hat{\vec{p}} - \frac{q}{c}\vec{A}$ describes a charged particle in an external electromagnetic field and is invariant under gauge transformation. Show that for $\hat{\vec{v}}$ the following commutation relations hold:

$$[\hat{x}_i, m\hat{v}_j] = i\hbar\delta_{ij} \tag{1}$$
$$[m\hat{v}_i, m\hat{v}_j] = i\hbar\frac{q}{c}\epsilon_{ijk}B_k, \tag{2}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$.

Problem 33. *Landau levels - alternative access*
Let $\vec{B} = (0, 0, B)$ be a constant, homogeneous magnetic filed in z-direction. Let the electric field and the scalar potential be zero, so that the vector potential $\vec{A}$ is a function of space only. Show that the Hamilton operator

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - \frac{q}{c}\vec{A}\right)^2 = \hat{H}_\perp + \hat{H}_\parallel \tag{3}$$

can be separated in a transversal and longitudinal part

$$\hat{H}_\perp = \frac{m}{2}\left(\hat{v}_x^2 + \hat{v}_y^2\right), \qquad \hat{H}_\parallel = \frac{m}{2}\hat{v}_z^2. \tag{4}$$

Show that both parts commute, i.e. $[\hat{H}_\perp, \hat{H}_\parallel] = 0$. Define the following operators

$$\hat{Q} = \frac{\sqrt{m/M}}{\omega_c}\hat{v}_x, \qquad \hat{P} = \sqrt{mM}\hat{v}_y, \tag{5}$$

where $\omega_c = qB/mc$ is the cyclotron frequency. Show that $\hat{Q}$ and $\hat{P}$ obey $[\hat{Q}, \hat{P}] = i\hbar$. Express $\hat{H}_\perp$ through $\hat{Q}$ and $\hat{P}$ and determine the eigenvalues of $\hat{H}_\perp$. What are the energy eigenvalues of $\hat{H}$ (*Landau levels*)?

♠ **Problem 34.** *Spin precession* (6 points)
Consider a spin-1/2 particle in an external magnetic field $\vec{B}(t) = (0, 0, B(t))$. Neglecting the motion of a particle the Hamiltonian is

$$\hat{H} = \frac{2\mu}{\hbar}\vec{B}(t) \cdot \hat{\vec{S}}. \tag{6}$$

At time $t = 0$ the state of the particle is given by the two-component vector

$$\chi(0) = \alpha\chi_+ + \beta\chi_-, \tag{7}$$

where $\chi_\pm$ are the eigenstates of $\hat{S}_z$ with eigenvalues $\pm\hbar/2$. Calculate the expectation values of $\langle\hat{S}_x(t)\rangle$, $\langle\hat{S}_y(t)\rangle$ and $\langle\hat{S}_z(t)\rangle$.

♠ **Problem 35.** *magnetic resonance* (8 points)
Consider again a spin-1/2 particle in an external magnetic field as in problem 34. Now consider $B(t) = (-B_\perp \cos(\omega t), B_\perp \sin(\omega t), B_\parallel)$. The state of the particles at time t can be expressed by

$$\chi(t) = \alpha(t)\chi_+ + \beta(t)\chi_-. \tag{8}$$

(a) Show that the Schrödinger equation for $\chi(t)$ with the Hamiltonian $\hat{H}$ from Eq. (6) can be expressed as

$$\frac{d}{dt}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = -i\begin{pmatrix}\Omega_\parallel & \Omega_\perp e^{i\omega t}\\ \Omega_\perp e^{-i\omega t} & -\Omega_\parallel\end{pmatrix}\cdot\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}, \tag{9}$$

with $\hbar\Omega_\parallel = \mu B_\parallel$ and $\hbar\Omega_\perp = \mu B_\perp$.

(b) Solve Eq. (9) with the initial conditions $\alpha(0) = 1, \beta(0) = 0$.

(c) At which frequency ω and at which time is $\chi(t) \propto \chi_-$?