Problem 1. – *Schrödinger representation of Klein-Gordon equation*
Show, that the functions φ and χ, which are defined by

$$\psi = \varphi + \chi, \quad i\hbar\frac{\partial}{\partial t}\psi = m_0 c^2 (\varphi - \chi)$$

fulfil the following equation:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \widehat{H}\begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{1}$$

where

$$\widehat{H} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\frac{\widehat{\overrightarrow{p}}^2}{2m_0} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} m_0 c^2.$$

Problem 2. – *Solutions of Klein-Gordon equation*
Show, that the following wave functions are solutions of equation (1) for negative or positive frequencies resp.:

$$\Psi_{(+)} = \int d^3p A(\overrightarrow{p})\begin{pmatrix} m_0c^2 + \hbar\omega_p \\ m_0c^2 - \hbar\omega_p \end{pmatrix} e^{i\left(\frac{\overrightarrow{p}\cdot\overrightarrow{x}}{\hbar} - \omega_p t\right)}$$

$$\Psi_{(-)} = \int d^3p A(\overrightarrow{p})\begin{pmatrix} m_0c^2 - \hbar\omega_p \\ m_0c^2 + \hbar\omega_p \end{pmatrix} e^{i\left(\frac{\overrightarrow{p}\cdot\overrightarrow{x}}{\hbar} + \omega_p t\right)}$$

where $\hbar\omega_p = c\sqrt{p^2 + m_0^2 c^2}$. Show, that the non relativistic limit yields:

$$\begin{pmatrix} \varphi_{(+)} \\ \chi_{(+)} \end{pmatrix} \to \frac{1}{\sqrt{L^3}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\left(\frac{\overrightarrow{p}\cdot\overrightarrow{x}}{\hbar} - \omega_p t\right)},$$

$$\begin{pmatrix} \varphi_{(-)} \\ \chi_{(-)} \end{pmatrix} \to \frac{1}{\sqrt{L^3}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\left(\frac{\overrightarrow{p}\cdot\overrightarrow{x}}{\hbar} + \omega_p t\right)}.$$

Problem 3. – *Charged Klein-Gordon particle*
Consider a Klein-Gordon particle in a Coulomb potential

$$V(\overrightarrow{r}) = V(r) = -\frac{Ze^2}{r}.$$

Show, that the stationary Klein-Gordon equation for the energy E can be written as

$$\left[(E - V(r))^2 - m_0^2 c^4 + \hbar^2 c^2 \Delta\right]\psi = 0.$$

Hint: Separate the radial and angular parts and derive an equation for the radial part.

Problem 4. – *α and β matrices*
Show, that the 4×4 matrices

$$\alpha_m = \begin{pmatrix} 0 & \sigma_m \\ \sigma_m & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

fulfil the following relations:

$$\{\alpha_m, \alpha_n\} = \alpha_m\alpha_n + \alpha_n\alpha_m = 2\delta_{mn}\mathbb{1}_4,$$

$$\{\alpha_m, \beta\} = \alpha_m\beta + \beta\alpha_m = 0,$$

$$\alpha_m^2\beta^2 = \mathbb{1}_4,$$

where σ_m are the Pauli matrices and $\mathbb{1}_d$ is the d-dimensional identity matrix.