Problem 1. – *Phase damping of a harmonic oscillator*
A coupling of a harmonic oscillator to a reservoir of oscillators that does not exchange excitations but leads to fast fluctuations of the oscillator energy results in the following Lindblad equation

$$\frac{d}{dt}\rho = \frac{1}{2}\Gamma\Big(2\hat{n}\,\rho\,\hat{n} - \hat{n}^2\,\rho - \rho\,\hat{n}^2\Big),$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the number operator, and $\Gamma > 0$. The above equation describes a *dephasing.*

(a) Find the general time-dependent solution of the equation in the occupation number basis, i.e.

$$\rho(t) = \sum_{n,m} \rho_{n,m}(t)\,|n\rangle\langle m|.$$

b) Consider as an initial state a so-called "cat" state:

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |n\rangle\Big).$$

Discuss your result.

c) Consider now a different "cat" state:

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\Big(|\alpha\rangle + |-\alpha\rangle\Big),$$

where $|\alpha\rangle$ is a coherent state, i.e. $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.

Problem 2. – *Conservation of positivity*
Show that if at $t = 0$ the density matrix is positive semi-definite it remains so after a small time interval Δt if the time evolution is governed by a Lindblad equation, i.e. that $\rho(\Delta t) \geq 0$ if $\rho(0) \geq 0$. Remember that an operator $\hat{A}$ is called positive semi-definite if

$$\langle\phi|\hat{A}|\phi\rangle \geq 0, \qquad \text{for all states} \quad |\phi\rangle.$$

Hint: Expand $\rho(\Delta t)$ in lowest order in Δt and use the Lindblad equation.

Problem 3. – *Infinite-temperature steady state*
Consider a spin 1/2 system, whose dynamics is governed by a Lindblad equation with arbitrary Hamiltonian and only *hermitian* Lindblas operators

$$\hat{L}_\mu^\dagger = \hat{L}_\mu.$$

Show that the infinite-temperature state is a stationary state of the open system dynamics.