Problem 1. – *Hard-sphere potential*
Consider the hard-sphere potential

$$V(r) = \begin{cases} \infty & \text{for} \quad r \leq r_0 \\ 0 & \text{for} \quad r > r_0. \end{cases}$$

The stationary solutions of the Schrödinger equation take the form

$$\varphi_k = \sum_{l=0}^{\infty} c_l R_{kl}(r) P_l(\cos\theta),$$

where $R_{kl}(r)$ fulfills the radial part of the Schrödinger equation for $r > r_0$

$$\left[\frac{1}{r}\frac{d^2}{dr^2}r - \frac{l(l+1)}{r^2} + k^2\right] R_{kl}(r) = 0 \tag{1}$$

for $E = \hbar^2 k^2/2m$ and vanishes for $r \leq r_0$

(a) Show that the general solution of eq.(1) has the form

$$R_{kl}(r) = B_l\Big[\cos(\delta_l)\, j_l(kr) - \sin(\delta_l)\, n_l(kr)\Big]$$

where $j_l(kr)$ and $n_l(kr)$ denote the spherical Bessel- and Neumann-functions and δ_l denotes the scattering phase of the l-th partial wave.

(b) Give an expression for the scattering phase δ_l.

(c) Find the s-wave scattering length a_0, as well as the scattering cross section in s-wave approximation.

Problem 2. – *Scattering amplitudes*
In the lecture we used the following expression to derive the connection between scattering amplitude and scattering phase:

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr)\, P_l(\cos\theta),$$

where j_l denotes the spherical Bessel-functions. Show, that this expression holds true. Use

$$j_l(x) = \frac{(-i)^l}{2}\int_{-1}^{1} d\xi\, e^{ix\xi}\, P_l(\xi).$$

Hint: Expand $e^{ikr\cos\theta}$ in Legendre polynomials $P_l(\cos\theta)$ using the completeness relation of the P_l. and determine the coefficients.

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Problem 3. – *Scattering of point particles*
Assume that the following scattering phases were found for point particles with mass m, energy $E = \frac{\hbar^2 k^2}{2m}$ and characteristic length r_0:

$$\tan \delta_l = -\frac{(r_0 k)^{2l+1}}{(2l+1)\left[(2l-1)!!\right]^2}$$

(a) Find a closed expression for the total cross section as a function of the energy E.

(b) For what energies E does the s-wave scattering provide a good approximation?