
Problem 1. – *quantized electric and magnetic field*

From the expression of the vector potential $\hat{\mathbf{A}}$ derived in class obtain expressions for the electric and magnetic fields in free space (use the Heisenberg picture)

$$\hat{\mathbf{E}} = -\frac{\partial \hat{\mathbf{A}}}{\partial t}, \qquad \hat{\mathbf{B}} = \nabla \times \hat{\mathbf{A}}.$$

Show that the fields are transversal and that for every mode $\mathbf{k}$ electric and magnetic component are orthogonal to each other and to $\mathbf{k}$ (transversality).

Problem 2. – *momentum of the quantized electromagnetic field*

The momentum of the quantized electromagnetic field is given by the integral over the momentum density

$$\hat{\mathbf{P}} = \varepsilon_0 \int d^3 r \, \hat{\mathbf{E}} \times \hat{\mathbf{B}}.$$

Show that $\hat{\mathbf{P}}$ can be expressed in the following form

$$\hat{\mathbf{P}} = \sum_{\mathbf{k},\lambda} \hbar \, \mathbf{k} \, \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda},$$

which can be interpreted as the total momentum of photons each with momentum $\hbar \, \mathbf{k}$.

Problem 3. – *angular momentum of the quantized electromagnetic field*

The angular momentum of the quantized electromagnetic field in Coulomb gauge (with respect to the coordinate origin $\mathbf{r}_0 = 0$) is given by the integral

$$\hat{\mathbf{L}}_{\text{tot}} = \varepsilon_0 \int d^3 r \, \mathbf{r} \times \Big(\hat{\mathbf{E}} \times \hat{\mathbf{B}} \Big).$$

Show that this expression can be decomposed into two terms

$$\begin{aligned} \hat{\mathbf{L}}_{\text{tot}} &= \hat{\mathbf{S}} + \hat{\mathbf{L}}_{\text{orb}} \\ &= \varepsilon_0 \int d^3 r \, \hat{\mathbf{E}} \times \hat{\mathbf{A}} + \varepsilon_0 \int d^3 r \, \sum_j \hat{E}_j (\mathbf{r} \times \nabla) \hat{A}_j. \end{aligned}$$

The first term does not contain $\mathbf{r}$ and is thus independent on the choice of the coordinate origin. It can be associated with an internal angular momentum of photons, while the second term describes the orbital angular momentum. Show that the internal angular momentum (spin) corresponds to an amount of $1\hbar$ per photon.