Problem 29. *–Foldy-Wouthuysen*
Consider a free Dirac particle in momentum space under Foldy-Wouthuysen Transformation $\Phi = \hat{U}\Psi, \hat{U} = e^{i\hat{S}}$, where

$$\hat{S} = \frac{-i}{2m_0c}\beta\vec{\alpha}\vec{p}\, f\left(\frac{p}{m_0c}\right), \qquad p = |\vec{p}|$$

(a) Show: $\beta e^{-i\hat{S}} = e^{i\hat{S}}\beta$

(b) Show by expansion of $e^{2i\hat{S}}$ that the Hamiltonian of a free Dirac particle transforms under $H_\Phi = e^{i\hat{S}}He^{-i\hat{S}} = e^{2i\hat{S}}(c\vec{\alpha}\vec{p} + \beta m_0c^2)$ to

$$\hat{H}_\Phi = \beta[m_0c^2\cos(y\ f(y)) + cp\sin(y\ f(y))] + \frac{\vec{\alpha}\vec{p}}{p}[cp\cos(y\ f(y)) - m_0c^2\sin(y\ f(y))] \tag{1}$$

where $y = \frac{p}{m_0c^2}$.

(c) Show, that the second term in (1) vanishes for $f(y) = \frac{1}{y}\arctan(y)$.

Problem 30. *–Foldy-Wouthuysen, again*
Find the wave-function of free spinors with momentum p, energy projection $\lambda = \pm 1$ and helicity $\pm\frac{\hbar}{2}$

$$\Phi_{p,\pm 1,\pm\frac{1}{2}}(\vec{r})$$

in Foldy-Wouthuysen representation.