
Problem 20. –*Velocity operator*
Find the true single-particle-velocity-operator of a Dirac-particle $\left[\frac{\mathrm{d}\hat{\vec{x}}}{\mathrm{d}t}\right]$ and compare the result with

$$\frac{\mathrm{d}\hat{\vec{x}}}{\mathrm{d}t} = \frac{-i}{\hbar}\left[\hat{\vec{x}}, \hat{H}\right] = c\vec{\alpha}$$

What are the eigenvalues and eigenstates of the true single-particle-velocity-operator? Compare the results with a classical relativistic particle.

Problem 21. –*Sigma-matrices*
Calculate all 6 matrices $\sigma_{\mu\nu}$, that fulfil

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

where $\gamma^0 = \beta$ and $\gamma^j = \beta\alpha_j$.

Problem 22. –*Lorentz-transformation*
Show, that the transformation-operator for Lorentz-transformations of Dirac-spinors

$$S = e^{\frac{-i}{\hbar}\omega\sigma_{\mu\nu}I^{\mu\nu}}$$

fulfils

$$S^{-1} = \gamma_0 S^\dagger \gamma_0$$

Problem 23. –*Boost of a Dirac particle*
The following equation is an eigenstate of the Dirac-Hamiltonian for vanishing momentum $\vec{p} = 0$:

$$\Psi_1(\vec{r}, t) = \frac{1}{\sqrt{V}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{\frac{-i}{\hbar}m_0c^2t}$$

Start with Ψ_1 and construct a solution for finite momentum $\vec{p} = (p_x, 0, 0)$ by using a Lorentz-transformation and compare your result with the one from **Problem 18**.

Problem 24. –*Invariance under gauge transformation*
Show explicitly, that the Dirac-equation in an external electromagnetic field

$$i\hbar\frac{\partial}{\partial t}\Psi = \left\{c\vec{\alpha}(\vec{p} - \frac{e}{c}\vec{A}) - eA_0 + m_0c^2\right\}\Psi$$

is invariant under gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \frac{\partial\lambda(x)}{\partial x^\mu}$$

$$\Psi(x) \to \Psi'(x) = \Psi(x)e^{-i\frac{e\lambda(x)}{\hbar c}}$$