Problem 20. – Velocity operator
Find the true single-particle-velocity-operator of a Dirac-particle \[ \frac{d\hat{x}}{dt} \] and compare the result with \[ \frac{d\hat{x}}{dt} = -\frac{i}{\hbar} [\hat{x}, \hat{H}] = c\tilde{a} \]

What are the eigenvalues and eigenstates of the true single-particle-velocity-operator? Compare the results with a classical relativistic particle.

Problem 21. – Sigma-matrices
Calculate all 6 matrices \( \sigma_{\mu\nu} \), that fulfil
\[ \sigma_{\mu\nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}] \]
where \( \gamma^0 = \beta \) and \( \gamma^j = \beta \alpha_j \).

Problem 22. – Lorentz-transformation
Show, that the transformation-operator for Lorentz-transformations of Dirac-spinors
\[ S = e^{-\frac{i}{\hbar} \omega_{\mu\nu} I^{\mu\nu}} \]
fulfils
\[ S^{-1} = \gamma_0 S^\dagger \gamma_0 \]

Problem 23. – Boost of a Dirac particle
The following equation is an eigenstate of the Dirac-Hamiltonian for vanishing momentum \( \vec{p} = 0 \):
\[ \Psi_1(\vec{r}, t) = \frac{1}{\sqrt{V}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar} m_0 c^2 t} \]

Start with \( \Psi_1 \) and construct a solution for finite momentum \( \vec{p} = (p_x, 0, 0) \) by using a Lorentz-transformation and compare your result with the one from Problem 18.

Problem 24. – Invariance under gauge transformation
Show explicitly, that the Dirac-equation in an external electromagnetic field
\[ i\hbar \frac{\partial}{\partial t} \Psi = \left\{ c\tilde{a}(\vec{p} - \frac{e}{c}\vec{A}) - eA_0 + m_0c^2 \right\} \Psi \]
is invariant under gauge transformation
\[ A_\mu \rightarrow A'_\mu = A_\mu + \frac{\partial \lambda(x)}{\partial x^\mu} \]
\[ \Psi(x) \rightarrow \Psi'(x) = \Psi(x)e^{-i\frac{\epsilon(\lambda(x))}{\hbar}} \]