Problem 12. –*Return of the Klein-Gordon equation*
Show, that the the functions φ and χ, which are defined by

$$\psi = \varphi + \chi \quad i\hbar\frac{\partial}{\partial t}\psi = m_0c^2(\varphi - \chi)$$

fulfil the following equation:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \hat{H}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \tag{1}$$

where

$$\hat{H} = \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\frac{\hat{\vec{p}}^2}{2m_0} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}m_0c^2$$

Problem 13. – *Solutions of the Klein-Gordon equation*
Show, that the following wave functions are solutions of equation (1) for negative or positive frequency resp.:

$$\Psi_{(+)} = \int \mathrm{d}^3p\ A(\vec{p})\begin{pmatrix}m_0c^2 + \hbar\omega_p\\ m_0c^2 - \hbar\omega_p\end{pmatrix}e^{i(\frac{\vec{p}\cdot\vec{x}}{\hbar} - \omega_p t)}$$
$$\Psi_{(-)} = \int \mathrm{d}^3p\ \tilde{A}(\vec{p})\begin{pmatrix}m_0c^2 - \hbar\omega_p\\ m_0c^2 + \hbar\omega_p\end{pmatrix}e^{i(\frac{\vec{p}\cdot\vec{x}}{\hbar} + \omega_p t)}$$

where $\hbar\omega_p = c\sqrt{p^2 + m_0^2c^2}$.
Show, that the non relativistic limit yields:

$$\begin{pmatrix}\varphi_{(+)}\\ \chi_{(+)}\end{pmatrix} \to \frac{1}{\sqrt{L^3}}\begin{pmatrix}1\\ 0\end{pmatrix}e^{i(\frac{\vec{p}\cdot\vec{x}}{\hbar} - \omega_p t)}$$
$$\begin{pmatrix}\varphi_{(-)}\\ \chi_{(-)}\end{pmatrix} \to \frac{1}{\sqrt{L^3}}\begin{pmatrix}0\\ 1\end{pmatrix}e^{i(\frac{\vec{p}\cdot\vec{x}}{\hbar} + \omega_p t)}$$

Problem 14. –*Charged Klein-Gordon particle*
Consider a Klein-Gordon particle in a Coulomb potential

$$V(\vec{r}) = V(r) = \frac{-Ze^2}{r}$$

Show, that the stationary Klein-Gordon equation for the energy E can be written as

$$\left[(E - V(r))^2 - m_0^2c^4 + \hbar^2c^2\Delta\right]\psi = 0$$

Hint: Separate the radial and angular parts and derive an equation for the radial part.

Problem 15. – *Gamma matrices*
Show, that the 4×4 gamma matrices

$$\alpha_m = \begin{pmatrix} 0 & \sigma_m \\ \sigma_m & 0 \end{pmatrix} , \qquad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

fulfil the following relations:

$$\begin{gathered} \{\alpha_m, \alpha_n\} = \alpha_m \alpha_n + \alpha_n \alpha_m = 2\delta_{mn} \mathbb{1}_4 \\ \{\alpha_m, \beta\} = \alpha_m \beta + \beta \alpha_m = 0 \\ \alpha_m^2 \beta^2 = \mathbb{1}_4 \end{gathered}$$

where σ_m are the Pauli matrices and $\mathbb{1}_d$ is the d-dimensional identity matrix.