Problem 5. – *Hard-sphere potential*
Assume the hard-sphere potential

$$V(r) = \begin{cases} \infty & \text{if } r \leq r_0 \\ 0 & \text{if } r > r_0 \end{cases}$$

The stationary solutions of the Schrödinger equation take the form

$$\varphi_k = \sum_{l=0}^{\infty} c_l R_{kl}(r) P_l(\cos\theta)$$

where $R_{kl}(r)$ fulfils the radial equation ($E = \frac{\hbar^2 k^2}{2m}$)

$$\left[\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2} r - \frac{l(l+1)}{r^2} + k^2\right] R_{kl}(r) = 0 \tag{1}$$

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(a) Show, that the general solution of (1) can be written in the form

$$R_{kl}(r) = B_l \left[\cos(\delta_l) j_l(kr) - \sin(\delta_l) n_l(kr)\right]$$

where $j_l(kr)$ and $n_l(kr)$ denote the spherical Bessel- and Neumann-functions and δ_l denotes the scattering phase of the l-th partial wave.

(b) Give an expression for the scattering phase δ_l.

(c) Find the s-wave scattering length a_0, as well as the scattering cross section in s-wave approximation.

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Problem 6. – *Scattering amplitudes*
In the lecture we used the following expression to derive the connection between scattering amplitude and scattering phase:

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos\theta)$$

where j_l denotes the spherical Bessel-functions. Show, that this expression holds true. Use

$$j_l(x) = \frac{(-i)^l}{2} \int_{-1}^{1} \mathrm{d}\xi e^{ix\xi} P_l(\xi)$$

Hint: expand $e^{ikr\cos\theta}$ in spherical harmonics and determine the coefficients.

Problem 7. – *Scattering of point particles*
One can find the following scattering phases of point particles with mass m and energy $E = \frac{\hbar^2 k^2}{2m}$, if scattered by a scattering centre of characteristic length r_0:

$$\tan \delta_l = \frac{-(r_0 k)^{2l+1}}{(2l+1)[(2l-1)!!]^2}$$

(a) Find a closed expression for the total cross section as a function of the energy E.

(b) For what energies E does the s-wave scattering provide a good approximation?