7.2 Superconducting qubits

Scalable quantum computing most likely means to go to the solid state. A promising avenue are superconducting qubits and Josephson elements.

- Solid-state devices are typically dissipative (origin lasers)
  \[\Rightarrow\] superconducting circuit

- Transition frequencies in superconducting qubits (SCQ) are in the range of 5-20 GHz; to avoid thermal fluctuations and noise
  \[\Rightarrow\] low temperatures

- To create systems with two relevant states (qubit), we need non-linear elements
  \[\Rightarrow\] Josephson junctions

Two types of qubits

(i) Charge qubit

(ii) Flux qubit
(A) quantum LC oscillator

Consider quantum description of an LC circuit

\[ V \quad C \quad L \quad I_L \quad V \text{ voltage at capacitor} \]
\[ I_L \quad I \text{ current} \]
\[ Q \quad \text{charge on capacitor} \]
\[ \Phi \quad \text{magn. flux through inductor} \]

Elements of circuit.

(13) \[ I_c = \frac{dQ}{dt} = C \frac{dV_c}{dt} \quad \Phi = \frac{\Phi}{L} \]

(14) \[ V_L = \frac{d\Phi}{dt} \]

Kirchhoff's laws: \[ V_L = V_c \quad I_c + I_L = 0 \]

(15) \[ \frac{d^2\Phi}{dt^2} + \frac{\Phi}{LC} = 0 \]

Harmonic oscillator

(16) \[ \omega_{LC} = \frac{1}{\sqrt{LC}} \]
energy of capacitor and of inductor

\[ H = \frac{1}{2C} Q^2 + \frac{1}{2L} \Phi^2 \]

Hamiltonian

\[ H = \frac{1}{2C} Q^2 + \frac{1}{2L} \Phi^2 \]

\[ E_n = \hbar \omega L_c (n + \frac{1}{2} ) \]

canonical coordinates: \( \Phi \)

momentum: \( Q \)

\[ \hat{\Phi}, \hat{Q} = \imath \hbar \]

(B) Josephson element

harmonic oscillator has equidistant spectrum. In order to obtain a qubit with two relevant states we need a nonlinear element \( \Rightarrow \) Josephson tunnel junction

\[ \Phi \]

supercconducting ring

\[ \Phi \]

flux through ring

\[ I_J \]

current through barrier

Analogously to a Bose-Einsenstein condensate the Cooper pairs of a superconductor are described by a complex wave function

\[ \Psi = 12^1 e^{i\Phi} \]
If a magnetic flux $\Phi$ penetrates the loop, the Cooper pairs pick up a phase proportional to the flux when going around the loop.

$$\Theta = \Delta \Phi = \frac{2e}{h} \frac{\Phi}{\Phi_0} = 2\pi \frac{\Phi}{\Phi_0}$$

$\Phi_0 = \hbar/2e$ flux quantum.

The tunnel current increases with increasing flux but must be periodic in $\Theta \mod 2\pi$.

$$I_J = I_0 \sin \Theta \quad \text{I}_0 \text{ critical Josephson current}$$

The loop acts as a primitive coil and has a capacity. The equivalent circuit diagram is thus:

![Circuit Diagram]

If there is an addition to the flux $\Phi$ created by the oscillating current an external magnetic flux $\Phi_e$ (used for tuning purposes):

$$I_L = \frac{\left(\Phi - \Phi_e\right)}{L} = \frac{5}{2eL} \left(\Theta - \Theta_e\right)$$

Kirkhoff yields then $I_C + I_L + I_J = 0$.

$$C \frac{d^2 \Phi}{dt^2} + I_0 \sin \left(2\pi \frac{\Phi}{\Phi_0}\right) + \Phi - \Phi_e = 0$$
or equivalently

$$
(25) \quad \frac{5}{2e} C \Theta + I_0 \sin \Theta + \frac{5}{2eL} (\Theta - \Theta_e) = 0
$$

It is easy to see that this is the Euler-Lagrange equation

$$
\frac{\partial}{\partial \Theta} \frac{\partial \mathcal{L}}{\partial \Theta'} - \frac{\partial \mathcal{L}}{\partial \Theta} = 0
$$

for

$$
(26) \quad \mathcal{L} = \frac{5}{4eC} \Theta^2 - E_J (1 - \cos \Theta) - E_L (\Theta - \Theta_e)^2
$$

where

$$
(27) \quad E_C = \frac{(2e)^2}{2C} \quad \text{charging energy}
$$

= energy of capacitor C with charge difference of one Cooper pair 2e

and

$$
(28) \quad E_J = \frac{5}{2e} I_0 \quad \text{Josephson energy}
$$

and

$$
(29) \quad E_L = \frac{\Phi_0^2}{4\pi^2 L} \quad \text{induction energy} \quad \text{(one flux quantum)}
$$

Hamiltonian

$$
H = p \Theta - \mathcal{L}
$$

with

$$
(30) \quad p = \frac{\partial \mathcal{L}}{\partial \Theta'} = \left( \frac{5}{2e} \right)^2 C \Theta
$$
Since \( \Theta = \frac{2e}{\hbar} \Phi \) (see eq. (21)) and \( \frac{\hbar}{2e} = V_L \) one finds

\[
(31) \quad p = \frac{\hbar}{2e} \quad CV = \frac{\hbar}{2e} \quad = \hbar \eta
\]

where \( \eta \) is the number of Cooper pairs

\[
(32) \quad H = E_c \hat{n}^2 + E_g (1 - \cos \Theta) + E_L \frac{(\Theta - \Theta_e)^2}{2}
\]

fundamental commutator

\[
(33) \quad [\hat{\Theta}, \hat{n}] = i
\]

Potential

\[
U = E_g (1 - \cos \Theta) + E_L (\Theta - \Theta_e)^2/2
\]

\( \Theta_e = 0 \)

\( \Theta_e = \pi/2 \)

If \( E_J > E_L \)
(c) Single cooper-pair box (SCB): charge qubit

Application of external voltage $V$ via gate capacitance allows to change number of cooper pairs: $E_L \approx 0$

$$H = E_c (\hat{n} - n_g)^2 - E_J \cos \Theta$$

Now
$$E_c = \frac{(2e)^2}{2(C_g + C_j)}$$
$$2e n_g = C_g V$$

$E_c \gg E_J$

Introduce basis states of number of cooper pairs $n$

$$H = \sum_n E_c (n - n_g)^2 |n\rangle \langle n|$$

$$(35) \quad - \frac{1}{2} E_J (|n\rangle \langle n+1| + |n+1\rangle \langle n|)$$

as
$$\langle n | \cos \Theta | n' \rangle = \delta_{n,n'+1}$$

Energy sequence of parabolas as function of $n_g$ plus small perturbation $\sim E_J$
for $n_g \approx \frac{1}{2}$ effective two-level system

\begin{equation}
H = -\frac{1}{2} B_z \delta z - \frac{1}{2} B_x \delta x
\end{equation}

\begin{equation}
B_z = E_c \left(1 - 2n_g\right) \quad B_x = E_J
\end{equation}

Varying $n_g$ through the applied voltage allows to perform single-bit operations

* quantum gates

parallel tunnel junctions with capacitive coupling

\begin{equation}
E_c' = \frac{(2e)^2}{2C}
\end{equation}

changing energy of coupling capacitor

\begin{equation}
H_{int} = \sum_{n_1, n_2} E_c' \left(n_{1} - n_{2}\right)^2 \langle n_1, n_2 \rangle \langle n_1, n_2 \rangle
\end{equation}
for \( n_1 \approx n_2 \approx \frac{1}{2} \) \( \langle n_1, n_2 \rangle \in \{00\rangle, 10\rangle, 11\rangle, 01\rangle, 10\rangle \}

\[
H_{int} = E_c' \left( |10\rangle\langle 01| + |10\rangle\langle 10| \right)
\]

\[
= E_c' \left( (1 + \delta_2^7)(1 - \delta_2^2) + (1 - \delta_2^7)(1 + \delta_2^2) \right)
\]

\[
(40) \quad H_{int} = E_c' \left( 2 - \delta_2^7 \delta_2^2 \right)
\]

Using interaction

**D. Josephson flux qubit**

\[ E_J \gg E_c \]

\[ E_J \quad \Phi \quad L \]

\[ C_J \]

Now \( E_L \) cannot be ignored

\[
(41) \quad H = E_c \hat{n}^2 + E_J (1 - \cos \Theta) + \frac{E_L}{2} (\Theta - \Theta_e)^2
\]

\[ S \quad (32) \]

Choose \( \Theta_e = \frac{\pi}{2} \) i.e. \( \Phi_e = \Phi_0 / 2 \)

\[
\begin{array}{c}
\text{"0"} \quad \text{"1"} \quad \text{"0"} \\
0 \quad \frac{\pi}{2} \quad 1 \quad \Phi_e \Phi_0 = \Theta_e / 2\pi
\end{array}
\]

\[ \frac{E_L}{2} \left( \Theta - \frac{\pi}{2} \right)^2 \]
Single-qubit Hamiltonian

\begin{equation}
(42) \quad H = -\frac{1}{2} B_2 \delta_2^2 - \frac{1}{2} B_x \delta_x
\end{equation}

Here \( B_2 = 4\pi \sqrt{6} (\beta L - 1) E_J \left( \frac{\Phi_e}{\Phi_0} - 1 \right) \)

\begin{equation}
(43) \quad \beta_L = E_J / (\Phi_0^2 / 4\pi^2 L)
\end{equation}

Two qubit gates can be realized via an inductive coupling of

\begin{figure}
\centering
\includegraphics[width=0.3\textwidth]{qubit_circuit.png}
\caption{Qubit circuit diagram}
\end{figure}

flux qubit 1  flux qubit 2

\textbf{end}