2. entangled states

2.1. The thougt experiment by Einstein, Podolski and Rosen (1935) EPR

1935 EPR put forward an argument that the quantum theory of Bohr, Sommerfeld and Heisenberg is either incomplete or that locality in the strict meaning of the words has to be given up. To this end EPR used a pair of particles A+B with position an momentum operators $\hat{x}_A, \hat{x}_B$, and $\hat{p}_A, \hat{p}_B$ in an entangled state where the difference in coordinates $\hat{x}_- = \hat{x}_A - \hat{x}_B$ and the sum of momenta $\hat{p}_+ = \hat{p}_A + \hat{p}_B$ are simultaneously well defined. This is possible since

(1) $$[\hat{x}_-, \hat{p}_+] = [\hat{x}_A - \hat{x}_B, \hat{p}_A + \hat{p}_B] = i\hbar - i\hbar = 0$$

We will discuss here however the variant suggested by Bohm 1951:

2 spin ½ particle in total $S=0$ state

$$A \longleftarrow \circ\circ \longrightarrow B$$

$$S=0$$

(2) $$|EPR\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow_z\rangle_A |\uparrow_z\rangle_B - |\uparrow_z\rangle_A |\downarrow_z\rangle_B\right)$$ entangled state

$|\downarrow_z\rangle$ eigenstate of σ_z with EV: -1

measurement of $\hat{\sigma}_z$ in A determines immediately value of measurement of $\hat{\sigma}_z$ in B with 100% certainty. This is however <u>not</u> a genuine quantum property. The same holds for classically correlated pairs, e.g. if one has two marbles one red, one green in a black box. Taking one out of the box tells the observer immediately the color of the other one.

<u>however</u> it is also true that

$$
\begin{aligned}
|EPR\rangle &= \frac{1}{\sqrt{2}}\Big(|\downarrow_x\rangle_A |\uparrow_x\rangle_B - |\uparrow_x\rangle_A |\downarrow_x\rangle_B \Big) \\
&= \frac{1}{\sqrt{2}}\Big(|\downarrow_y\rangle_A |\uparrow_y\rangle_B - |\uparrow_y\rangle_A |\downarrow_y\rangle_B \Big)
\end{aligned} \tag{3}
$$

i.A. measurement of <u>any</u> spin direction (σ_x, σ_y or σ_z) in A determines immediately and with certainty the corresponding value in B!

<u>EPR paradox</u> assuming that causality holds than

- either QM is <u>non-local</u>: since $[\sigma_x^B, \sigma_y^B] \neq 0$ the spin in x and y direction cannot be simultaneously fixed; projection in A to either x or y direction determines immediately if measurement of x or y in B yields pre-determined result (while y or x would not)

- or QM is <u>incomplete</u> i.e. there are hidden parameter that do fix both σ_x and σ_y in B

Bell theorem (1964)

> if <u>local</u> hidden variable exist, then correlation measurements have to fulfill certain inequalities (Bell's inequalities)

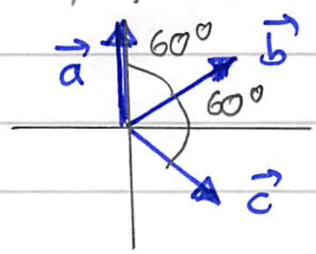
correlation of spin of A in direction $\vec{a}$ with spin in B in direction $\vec{b}$

$$(4) \quad E(\vec{a},\vec{b}) = \langle (\vec{\sigma}_A \cdot \vec{a})(\vec{\sigma}_B \cdot \vec{b}) \rangle$$

<u>Bell:</u>

$$(5) \quad \boxed{|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| \leq 1 + E(\vec{b},\vec{c})}$$

consider EPR state $|EPR\rangle$, eq. (2), and 3 directions in one plane



EPR state is isotrop (eq. (2), (3)), so choose one axis as "$\vec{e}_z$" $= \vec{a}$; $\vec{b}$ rotated

$$E(\vec{a},\vec{b}) = \langle EPR| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_A \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}_B |EPR\rangle$$

$$|EPR\rangle = \frac{1}{2}\left((1,0)_A (0,1)_B - (0,1)_A (1,0)_B \right)$$

$$= \frac{1}{2}\left[\begin{pmatrix}1\\0\end{pmatrix}_A^T \begin{pmatrix}0\\1\end{pmatrix}_B^T - \begin{pmatrix}0\\1\end{pmatrix}_A^T \begin{pmatrix}1\\0\end{pmatrix}_B^T\right] \begin{pmatrix}1&0\\0&-1\end{pmatrix}_A \begin{pmatrix}\cos\varphi & \sin\varphi\\ \sin\varphi & -\cos\varphi\end{pmatrix}_B$$

(6) $$\cdot \left[\begin{pmatrix}1\\0\end{pmatrix}_A \begin{pmatrix}0\\1\end{pmatrix}_B - \begin{pmatrix}0\\1\end{pmatrix}_A \begin{pmatrix}1\\0\end{pmatrix}_B\right]$$

$$= \frac{1}{2}\left\{ \begin{pmatrix}1\\0\end{pmatrix}_A^T \begin{pmatrix}1&0\\0&-1\end{pmatrix}_A \begin{pmatrix}1\\0\end{pmatrix}_A \cdot \begin{pmatrix}0\\1\end{pmatrix}_B^T \begin{pmatrix}\cos\varphi & \sin\varphi\\ \sin\varphi & -\cos\varphi\end{pmatrix}_B \begin{pmatrix}0\\1\end{pmatrix}_B \right.$$

$$\left. + \begin{pmatrix}0\\1\end{pmatrix}_A^T \begin{pmatrix}1&0\\0&-1\end{pmatrix}_A \begin{pmatrix}0\\1\end{pmatrix}_A \cdot \begin{pmatrix}1\\0\end{pmatrix}_B^T \begin{pmatrix}\cos\varphi & \sin\varphi\\ \sin\varphi & -\cos\varphi\end{pmatrix}_B \begin{pmatrix}1\\0\end{pmatrix}_B \right\}$$

$$= \frac{1}{2}\left(-\cos\varphi - \cos\varphi\right) = -\cos\varphi \qquad \varphi \sphericalangle (\vec{a}, \vec{b})$$

$\Rightarrow$

$$E(\vec{a},\vec{b}) = -\cos(60°) = -\frac{1}{2} = E(\vec{b},\vec{c})$$

$$E(\vec{a},\vec{c}) = -\cos(120°) = \frac{1}{2}$$

$\Rightarrow$

(7) $$|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| = 1 \quad > \quad 1 + E(\vec{b},\vec{c}) = \frac{1}{2}$$

$\boxed{|EPR\rangle \text{ leads to violation of Bell inequality !}}$

experiments (Clauser 1969, Aspect 1982) have proven violation of Bell inequalities $\Rightarrow$ no local hidden variables

Bell inequalities can only be violated by entangled states.
$\Rightarrow$ entangled states have strong non-classical properties

this becomes even more apparent in entangled 3-partite states

<u>Greenberger, Horn, Zeilinger (GHZ) thought experiment (1989)</u>

(8) $$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\uparrow\rangle_B|\uparrow\rangle_C - |\downarrow\rangle_A|\downarrow\rangle_B|\downarrow\rangle_C\right)$$

is eigenstate of

(9) $$\left.\begin{array}{l} \hat{\sigma}_y^A\,\hat{\sigma}_y^B\,\hat{\sigma}_x^C \\ \hat{\sigma}_y^A\,\hat{\sigma}_x^B\,\hat{\sigma}_y^C \\ \hat{\sigma}_x^A\,\hat{\sigma}_y^B\,\hat{\sigma}_y^C \end{array}\right\} \quad \text{eigenvalue } +1$$

is also eigenstate of

(10) $$\hat{\sigma}_x^A\,\hat{\sigma}_x^B\,\hat{\sigma}_x^C \qquad \text{eigenvalue } -1$$

this is opposite to classically correlated case

(11) $$\begin{array}{ll} S_y^A S_y^B S_x^C = +1 & \qquad (S_{x,y} = \pm 1) \\ S_y^A S_x^B S_y^C = +1 & \\ S_x^A S_y^B S_y^C = +1 & \end{array}$$

product:

(12) $$S_y^{A^2} S_x^A\, S_y^{B^2} S_x^B\, S_y^{C^2} S_x^C = S_x^A S_x^B S_x^C = +1$$

$\boxed{\text{entanglement is a genuine quantum property}}$

important (maximally) entangled states:

2-partite states: Bell states (w/o normalization)

(13) $|\phi_+\rangle = |00\rangle + |11\rangle \qquad |\psi_+\rangle = |01\rangle + |10\rangle$

$|\phi_-\rangle = |00\rangle - |11\rangle \qquad |\psi_-\rangle = |01\rangle - |10\rangle$

3-partite state (w/o normalization)

$$|GHZ\rangle = |000\rangle - |111\rangle$$

(14)

$$|W\rangle = |100\rangle + |010\rangle + |001\rangle$$

we will see that entanglement plays a central role in quantum information science

2.2 Dense coding: entanglement as ressource

Bennett & Wiesner 1992: Alice wants to send bits to Bob

$\boxed{A}$ $\qquad\qquad\qquad\qquad$ $\boxed{B}$

100101 $\qquad\qquad\qquad\qquad$ 100101

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad$ $\uparrow$

$|100101\rangle \xrightarrow{\text{send to B}} |100101\rangle$

prepare eigenstate

assume A, B share a Bell state $|\phi_+\rangle$

A can turn $|\phi_+\rangle$ by local operations into all other Bell states

$$(15)\quad \begin{aligned} |\phi_+\rangle &= |00\rangle + |11\rangle \\ |\phi_-\rangle &= |00\rangle - |11\rangle \\ |\psi_+\rangle &= |10\rangle + |01\rangle \\ |\psi_-\rangle &= |10\rangle - |01\rangle \end{aligned} \qquad \sigma_z^A, \quad \sigma_x^A, \quad \sigma_x^A \sigma_z^A$$

can encode 2 classical bits; sending of particle at A to B allows B to detect which of the 4 Bell states

2.3 Quantum teleportation

- Is there a way to transfer an unknown state in A to B without sending the particle (physical object)?
- neither measurement nor cloning works!

Bennett, Brassard, Crepeau, Josza, Peres & Wooters 1993:
(PRL 70, 1895 (1993))

assume Alice and Bob posses Bell-state

$$(16)\quad |\phi_+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

in addition Alice has a spin 1/2 system in an <u>unknown</u> state

$$(17)\quad |\psi_A\rangle = \alpha|0\rangle_A + \beta|1\rangle_A$$

total state

$$|\psi_A\rangle|\phi_+\rangle = \frac{1}{\sqrt{2}}\left(\alpha|0\rangle_A + \beta|1\rangle_A\right)\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

$$\begin{aligned}
\sim\ & \left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)_A \left(\alpha|0\rangle + \beta|1\rangle\right)_B \\
& + \left(|0\rangle|0\rangle - |1\rangle|1\rangle\right)_A \left(\alpha|0\rangle - \beta|1\rangle\right)_B \\
& + \left(|1\rangle|0\rangle + |0\rangle|1\rangle\right)_A \left(\alpha|1\rangle + \beta|0\rangle\right)_B \\
& + \underbrace{\left(|1\rangle|0\rangle - |0\rangle|1\rangle\right)_A}_{\text{4 orthogonal Bell states in A}} \left(-\alpha|1\rangle + \beta|0\rangle\right)_B
\end{aligned} \tag{18}$$

orthogonal states can be distinguished by measurements. E.g. measurement of $|\psi_+\rangle_A \sim |1\rangle|0\rangle + |0\rangle|1\rangle$ projects state to

$$\varrho_0 \rightarrow \frac{|\psi_+\rangle_A\langle\psi_+|\,\varrho_0\,|\psi_+\rangle_A\langle\psi_+|}{\mathrm{Tr}\left\{|\psi_+\rangle_A\langle\psi_+|\,\varrho_0\right\}} \;\hat{=}\; \alpha|1\rangle_B + \beta|0\rangle_B \tag{19}$$

Now Alice has just to tell Bob that she measured $|\psi_+\rangle$ and Bob applies σ_x^B

$$\sigma_x^B\left(\alpha|1\rangle_B + \beta|0\rangle_B\right) = \alpha|0\rangle_B + \beta|1\rangle_B \tag{20}$$

state has been transferred from A to B without ever looking at it! <u>quantum teleportation</u>

collapse of state is instantaneous
$\Rightarrow$ information transfer faster than light ?

no! Without information which state Alice measured Bob has no information encoded in his state since

(21) $\quad \rho_B = Tr_A\{\rho_0\} = \mathbb{1}$

teleportation : entangled state as ressource + LOCC operation

2.4 Criterion and quantitative measure of entanglement of pure bi-partite states

as we have seen $|\psi\rangle$ is entangled if it cannot be written as

$$|\psi\rangle = |\phi\rangle_A |\phi_B\rangle$$

i.e. necessary and sufficient condition for entanglement is that Schmidtnumber $S > 1$.

for separable state

(22)
$$\rho_A = Tr_B\{|\psi\rangle\langle\psi|\} = |\phi\rangle_{A\,A}\langle\phi| \quad \text{i.e. } \rho_A^2 = \rho_A$$

$$\rho_B = Tr_A\{|\psi\rangle\langle\psi|\} = |\phi_B\rangle\langle\phi_B| \quad \text{i.e. } \rho_B^2 = \rho_B$$

i.e. reduced density operator of each part is pure

$$\boxed{\begin{array}{c} |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \text{ separabel} \\ \Updownarrow \\ \text{Schmidt number } S = 1 \\ \Updownarrow \\ \text{reduced density operator} = \text{pure state} \end{array}}$$

criterion

quantitative measure of entanglement ?

should have the following properties

(23)
(i) $E \geq 0$
(ii) $E = 0$ for Schmidt number $S = 1$
(iii) E should be extensive (in sense of thermodynamics)
(iv) $E = \max$ if all coefficients of Schmidt decomposition equal

$\Rightarrow$

(24) $$\boxed{E(|\psi\rangle\langle\psi|) = -\,Tr_B\{\rho_B \ln \rho_B\} = -\,Tr_A\{\rho_A \ln \rho_A\}}$$

i.e. von-Neumann entropy of subsystem

rem: Schmidt decomposition $\dim(\mathcal{H}_A) = M \leq \dim(\mathcal{H}_B) = N$

$$|\psi\rangle = \sum_{j=1}^{M} c_j \, |j\rangle_A |j\rangle_B$$

$$\rho_B = \sum_{j=1}^{M} c_j^2 \, |j\rangle_{B\,B}\langle j| \qquad \rho_A = \sum_{j=1}^{M} c_j^2 \, |j\rangle_{A\,A}\langle j|$$

$$E(|\psi\rangle\langle\psi|) = -\,Tr_B\left\{\sum_{j=1}^{M} c_j^2 \, |j_B\rangle\langle j_B| \, \ln\left(\sum_{i=1}^{M} c_i^2 \, |i_B\rangle\langle i_B|\right)\right\}$$

$\Rightarrow$

(25) $$E = -\sum_{j=1}^{M} c_j^2 \ln c_j^2$$ fulfills (i)–(iv)

(26) $$E_{max} = \ln M$$ maximum entanglement entropy $\ln(\min(\dim \mathcal{H}_A, \mathcal{H}_B))$

One easily recognizes

> A separable state $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$ cannot be transformed to an entangled pure state by LOCC

2.5 Entanglement of mixed bi-partite states

A pure state

$$\rho = \left(|\phi_A\rangle|\phi_B\rangle\right)\left(\langle\phi_A|\langle\phi_B|\right) \quad \text{is separable}$$

We will still call ρ separable if it is a convex sum of separable pure states with probabilities p_j:

(27) $$\rho = \sum_{j=1}^{k} p_j \left(|j_A\rangle|j_B\rangle\right)\left(\langle j_A|\langle j_B|\right) \Leftrightarrow \underline{\underline{\text{separable !}}}$$

Otherwise ρ is called entangled. States of the form (27) can be generated by LOCC operations from pure separable states

> LOCC operations cannot turn separable states into entangled states

A necessary condition for separability is given by a <u>theorem of Peres</u> PRL $\underline{77}$, 1413 (1996)

(28) $\boxed{\text{if } \rho \text{ is separable} \Rightarrow \rho^{T_A} \geq 0 \text{ and } \rho^{T_B} = (\rho^{T_A})^T \geq 0}$

ρ^{T_A} is the partial transpose with respect to sub-system A

<u>proof:</u> ρ separable $\rho = \sum_{j=1}^{k} p_j \left(|j_A\rangle |j_B\rangle\right)\left(\langle j_A| \langle j_B|\right)$

$$\rho^{T_A} = \sum_{j=1}^{n} p_j \left(|j_A\rangle\langle j_A| \otimes |j_B\rangle\langle j_B|\right)^{T_A}$$

using $A^+ = (A^T)^*$ $\left(|i\rangle\langle i|\right)^+ = |i\rangle\langle i|$

$\Rightarrow$

$$\rho^{T_A} = \sum_{j=1}^{k} p_j \left(|j_A^*\rangle\langle j_A^*| \otimes |j_B\rangle\langle j_B|\right)$$

$$= \sum_{j=1}^{k} p_j \left(|j_A^*\rangle|j_B\rangle \langle j_A^*|\langle j_B|\right) \geq 0 \quad \square$$

M., P. and R. Horodecki have shown for $\mathcal{H}_A = \mathbb{C}^2$ and $\mathcal{H}_B = \mathbb{C}^2$ or $\mathcal{H}_B = \mathbb{C}^3$ Peres criterion is not only necessary but also sufficient

<u>Horodecki theorem</u>

(29) in $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$

$$\boxed{\rho \text{ separable} \Longleftrightarrow \rho^{T_A} \geq 0}$$

rem: a general practical necessary and sufficient condition for mixed-state entanglement is still not found

2.6 General entanglement measures: entanglement of destillation, entanglement of formation

first approach:

Suppose Alice and Bob have n copies of the same mixed bi-partite state ρ_{AB}. Suppose further they can generate out of these $k \leq n$ pure Bell states by LOCC operations. This corresponds to a destillation of a smaller number of maximally entangled pure states

(30) $$\boxed{E_D(\rho_{AB}) = \lim_{n\to\infty} \frac{k_{max}}{n}}$$ entropy of destillation

Second approach:

Suppose Alice and Bob have an arbitrary number of Bell states at their disposal. Suppose further they want to create n copies of ρ_{AB} by LOCC operations using k' Bell pairs. This corresponds to the creation of ρ_{AB} out of maximally entangled pure states

(31) $$\boxed{E_F(\varrho_{AB}) = \lim_{n\to\infty} \frac{k'_{min}}{n}}$$ entropy of formation

(30) and (31) are good definitions but practically useless.
For pure states one can show

(32) $$\boxed{E_D(|\psi\rangle\langle\psi|) = E_F(|\psi\rangle\langle\psi|) = E(|\psi\rangle\langle\psi|)}$$

where E is the von-Neumann entropy

For mixed states E is no longer a good measure since also mixed states of the form of eq. (27) have $E \neq 0$.

relative entropy of entanglement E_R

(33) $$\boxed{E_R(\varrho_{AB}) = \min_{\sigma \in \text{separabel}} Tr\{\varrho \ln \varrho - \varrho \ln \sigma\}}$$

One can show for mixed states

(34) $$\boxed{E_D(\varrho_{AB}) \le E_R(\varrho_{AB}) \le E_F(\varrho_{AB})}$$

rem: none of these measures are easily handible

concurrence

For a mixed state of two spin 1/2 systems the concurrence is a useful entanglement measure related to E_F

(35) $$\boxed{C(\varrho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)}$$

where $\lambda_1, \dots, \lambda_4$ are eigenvalues in decreasing order of

(36) $$R = \left(\sqrt{\varrho}\, \tilde{\varrho} \sqrt{\varrho}\right)^{1/2} \qquad \tilde{\varrho} = (\sigma_y \otimes \sigma_y)\, \varrho^* (\sigma_y \otimes \sigma_y)$$

logarithmic negativity

An upper bound to $E_D(\varrho)$ can be found from

(37) $$E_N(\varrho) = \log_2 \| \varrho^{T_A} \|$$

ϱ^{T_A} partial transpose and $\| X \| = \mathrm{Tr} \sqrt{X^+ X}$

bound entanglement

Mixed states that are non-separable, however with $E_D(\varrho) = 0$ are not destillable and are called bound entangled. Since they are not destillable they cannot be used as an entanglement ressource.

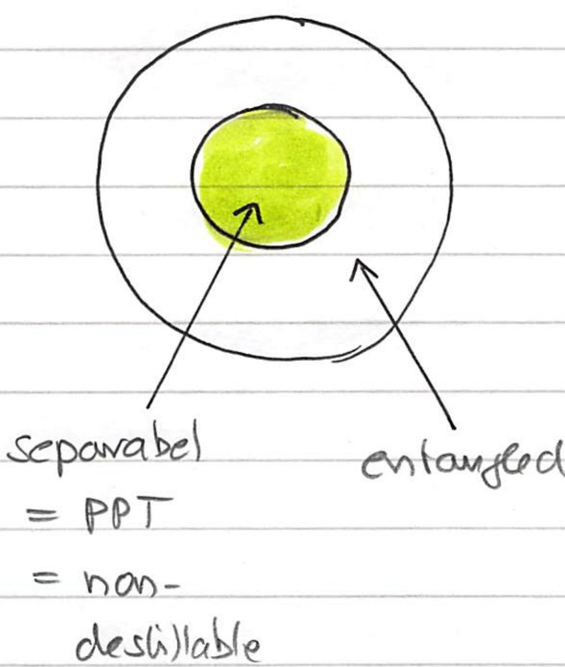
Horodecki's (PRL <u>80</u>, 5239 (1999)), Dür et al. (2000)

$$\boxed{\rho^{T_A} \geq 0 \quad \Longrightarrow \quad \rho \text{ is non destillable}}$$

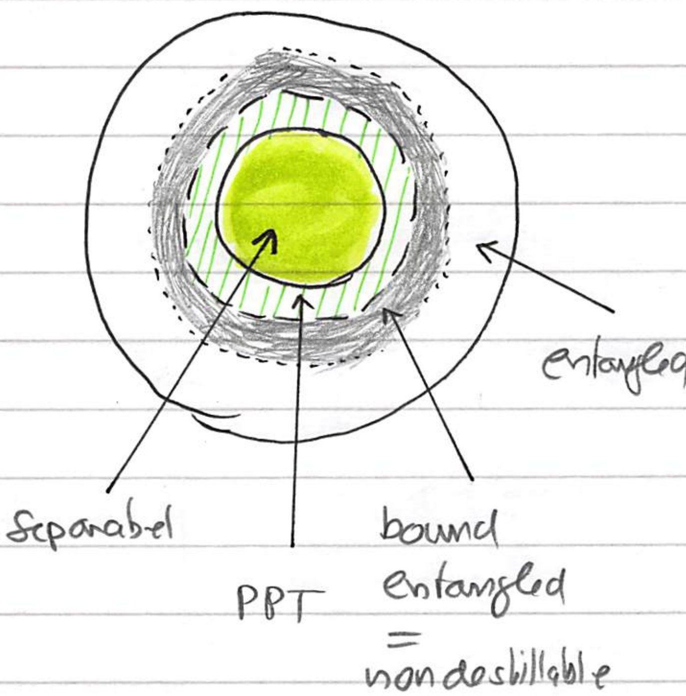
i.e. it is either separable or at best bound entangled

for $\mathbb{C}^2 \otimes \mathbb{C}^2$ also the inverse is true

<u>$\mathbb{C}^2 \otimes \mathbb{C}^2$</u>



<u>general</u>



PPT : positive partial transpose