2. Entangled states

2.1. The thought experiment by Einstein, Podolski, and Rosen (1935) EPR

1935 EPR put forward an argument that the quantum theory of Bohr, Sommerfeld and Heisenberg is either incomplete or that locality in the strict meaning of the words has to be given up. To this end EPR used a pair of particles \( A + B \) with position and momentum operators \( \hat{x}_A, \hat{x}_B \) and \( \hat{p}_A, \hat{p}_B \) in an entangled state where the difference in coordinates \( \hat{x}_- = \hat{x}_A - \hat{x}_B \) and the sum of momenta \( \hat{p}_+ = \hat{p}_A + \hat{p}_B \) are simultaneously well defined. This is possible since

\[
[\hat{x}_-, \hat{p}_+] = [\hat{x}_A - \hat{x}_B, \hat{p}_A + \hat{p}_B] = i\hbar - i\hbar = 0
\]

We will discuss here however the variant suggested by Bohm 1957:

A spin \( \frac{1}{2} \) particle in total \( S = 0 \) state

\[
A \leftarrow \begin{array}{c} 00 \\ S=0 \end{array} \rightarrow B
\]

\[
|EPR\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\uparrow\rangle_A |\downarrow\downarrow\rangle_B - |\uparrow\downarrow\rangle_A |\downarrow\uparrow\rangle_B \right) \quad \text{entangled state}
\]

\( |\uparrow\rangle \) eigenstate of \( \hat{S}_z \) with EV: -1
measurement of $\hat{d}_z$ in A determines immediately
value of measurement of $\hat{d}_z$ in B with 100% certainty.
This is however not a genuine quantum property.
The same holds for classically correlated pairs, e.g.
if one has two worlds one red $\bullet$, one green $\circ$ in
a black box. Taking one out of the box kills the observer
immediately the color of the other one.

However, it is also true that

$$|EPR\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow_x\rangle_A |\uparrow_x\rangle_B - |\downarrow_x\rangle_A |\downarrow_x\rangle_B \right)$$

$$= \frac{1}{\sqrt{2}} \left( |\uparrow_y\rangle_A |\uparrow_y\rangle_B - |\downarrow_y\rangle_A |\downarrow_y\rangle_B \right)$$

i.e. measurement of any spin direction ($\hat{d}_x$, $\hat{d}_y$ or
$\hat{d}_z$) in A determines immediately and with certainty
the corresponding value in B!

**EPR paradox** assuming that causality holds then

- either QM is non-local: since $[\hat{d}_x, \hat{d}_y] \neq 0$
  the spin in x and y direction cannot be simultane-
 ously fixed; projection in A to either x or y direction
determines immediately if measurement of x or y
in B yields pre-determined result (while y or x
would not)
- or QM is incomplete i.e. there are hidden parameters that do fix both dx and dy in B

**Bell theorem** (1964)

If local hidden variable exist, then correlation measurements have to fulfill certain inequalities (Bell’s inequalities)

Correlation of spin of A in direction $\vec{a}$ with spin in B in direction $\vec{b}$

$$E(\vec{a}, \vec{b}) = \langle (\vec{a} \cdot \hat{a}) (\vec{b} \cdot \hat{b}) \rangle$$

Bell:

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c})| \leq 1 + E(\vec{b}, \vec{c})$$

Consider EPR state $|EPR\rangle$, eq. (21), and 3 directions in one plane

\[ E(\vec{a}, \vec{b}) = \langle EPR | \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_A \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}_B |EPR \rangle \]

\[ |EPR\rangle = \frac{1}{2} \left( |1\rangle_A |0\rangle_B - |0\rangle_A |1\rangle_B \right) \]

EPR state is isotropic (eq. (21, 31)), so choose one axis and $\vec{c}_z = \vec{a}$; $\vec{b}$ rotated
\[
\frac{1}{2} \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{A}^{T} \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{B}^{T} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{A}^{T} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{B}^{T} \right] \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{A}^{T} \begin{pmatrix} \cos \varphi \sin \varphi \\ \sin \varphi - \cos \varphi \end{pmatrix}_{B}^{T} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix}_{A}^{T} \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{B}^{T}
\end{pmatrix}
\]

(6)

\[
\frac{1}{2} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{A}^{T} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{A}^{T} \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{A}^{T} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{B}^{T} \begin{pmatrix} \cos \varphi \sin \varphi \\ \sin \varphi - \cos \varphi \end{pmatrix}_{B}^{T} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{B}^{T} \right\}
\]

\[
+ \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{A}^{T} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{A}^{T} \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{A}^{T} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{B}^{T} \begin{pmatrix} \cos \varphi \sin \varphi \\ \sin \varphi - \cos \varphi \end{pmatrix}_{B}^{T} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{B}^{T} \right\}
\]

\[
= \frac{1}{2} \left( -\cos \varphi - \cos \varphi \right) = -\cos \varphi \quad \varphi \not\in (\pi, \pi)
\]

\[
\Rightarrow
\]

\[
E(\hat{a}, \hat{b}) = -\cos(60^\circ) = -\frac{1}{2} = E(\hat{b}, \hat{c})
\]

\[
E(\hat{a}, \hat{c}) = -\cos(120^\circ) = \frac{1}{2}
\]

\[
\Rightarrow
\]

\[
|E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c})| = 1 \quad > \quad 1 + E(\hat{b}, \hat{c}) = \frac{1}{2}
\]

\[\text{EPR} \text{ leads to violation of Bell inequality!}\]

Experiments (Clauser 1969, Aspect 1982) have proven violation of Bell inequalities \(\Rightarrow\) no local hidden variable.

Bell inequality can only be violated by entangled states. \(\Rightarrow\) entangled states have strong non-classical properties.
this becomes even more apparent in entangled 3-party states

Greenberger, Horne, Zeilinger (GHZ) thought experiment (1989)

\( |\text{GHZ}\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow_A \downarrow_B \downarrow_C \rangle - |\downarrow_A \uparrow_B \downarrow_C \rangle \right) \)

is eigenstate of

\[
\begin{bmatrix}
\hat{\sigma}_x^A & \hat{\sigma}_y^B & \hat{\sigma}_z^C \\
\hat{\sigma}_y^A & \hat{\sigma}_x^B & \hat{\sigma}_z^C \\
\hat{\sigma}_z^A & \hat{\sigma}_y^B & \hat{\sigma}_x^C \\
\end{bmatrix}
\]

\( \begin{array}{c}
\text{eigenvalue} +1 \\
\end{array} \)

is also eigenstate of

\( \hat{\sigma}_x^A \hat{\sigma}_x^B \hat{\sigma}_x^C \) \quad \text{eigenvalue} -1

this is opposite to classically correlated case

\[
\begin{align*}
\hat{S}_y^A \hat{S}_y^B \hat{S}_x^C &= +1 \\
\hat{S}_y^A \hat{S}_x^B \hat{S}_y^C &= +1 \\
\hat{S}_x^A \hat{S}_y^B \hat{S}_y^C &= +1 \\
\end{align*}
\]

\( S_{x,y} = \pm 1 \)

product:

\[
\begin{align*}
\hat{S}_y^A \hat{S}_x^A \hat{S}_y^B \hat{S}_x^B \hat{S}_y^C \hat{S}_x^C &= \hat{S}_x^A \hat{S}_x^B \hat{S}_x^C \\
&= +1 \\
\end{align*}
\]

entanglement is a genuine quantum property
important (maximally) entangled states:

2-particle states: Bell states (w/o normalization)

\( |\Phi^+\rangle = |10\rangle + |11\rangle \)
\( |\Phi^-\rangle = |10\rangle - |11\rangle \)

\( |\Psi^+\rangle = |10\rangle + |10\rangle \)
\( |\Psi^-\rangle = |10\rangle - |10\rangle \)

3-particle state (w/o normalization)

\( |\psi_{12} \rangle = |100\rangle - |111\rangle \)

\( |w\rangle = |100\rangle + |110\rangle + |001\rangle \)

We will see that entanglement plays a central role in quantum information science.

2.2 Dense coding: entanglement as resource

Bennett & Wiesner 1992: Ali wants to send bob to Bob

\[
\begin{array}{c}
|A\rangle \\
100 101 \\
\uparrow \\
|100 101\rangle \\
\end{array} \quad \begin{array}{c}
|B\rangle \\
100 101 \\
\uparrow \\
|100 101\rangle \\
\end{array}
\]

prepare eigenstate send to Bob

assume A, B share a Bell state \( |\Phi^+\rangle \)
A can turn $|\Psi_+\rangle$ by local operations into all other Bell states:

$$
|\Phi_+\rangle = |00\rangle + |11\rangle \\
|\Phi_-\rangle = |00\rangle - |11\rangle \\
|\Psi_+\rangle = |10\rangle + |01\rangle \\
|\Psi_-\rangle = |10\rangle - |01\rangle
$$

Can encode 2 classical bits: sending of particle $A$ to $B$ allows $B$ to detect which of the 4 Bell states.

### 2.3 Quantum Teleportation

- Is there a way to transfer an unknown state in $A$ to $B$ without sending the particle (physical object)?
- Neither measurement nor cloning works!

Bennett, Brassard, Crepeau, Jozsa, Peres & Wootters 1993. (PRL 70, 1895 (1993))

Assume Alice and Bob posses Bell-state

$$
(16) \quad |\Phi_+\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \right)
$$

In addition Alice has a spin 1/2 system in an unknown state

$$
(17) \quad |\psi_A\rangle = \alpha |0\rangle_A + \beta |1\rangle_A
$$
total state

\[ |2^+_A, 1^+_B\rangle = \frac{1}{\sqrt{2}} \left( \alpha |10\rangle_A + \beta |11\rangle_A \right) \left( |10\rangle_B + |11\rangle_B \right) \]

\[ \sim (|10\rangle_A + |11\rangle_A )_A (\alpha |10\rangle_B + \beta |11\rangle_B )_B \]

\[ + (|10\rangle_A - |11\rangle_A )_A (\alpha |10\rangle_B - \beta |11\rangle_B )_B \]

\[ + (|11\rangle_A + |10\rangle_A )_A (\alpha |11\rangle_B + \beta |10\rangle_B )_B \]

\[ + (|11\rangle_A - |10\rangle_A )_A (-\alpha |11\rangle_B + \beta |10\rangle_B )_B \]

4 orthogonal \( B \) states w/ \( A \)

Orthogonal states can be distinguished by measurement. E.g., measurement of \( |2^+_A\rangle \sim |11\rangle_A + |10\rangle_A \) projects state to

\[ |2^+_A\rangle \xrightarrow{\text{E}} \frac{|2^+_A\rangle + |2^-_A\rangle}{\sqrt{2}} \]

\[ \xrightarrow{\text{meas}} \alpha |11\rangle_B + \beta |10\rangle_B \]

Now Alice has just told Bob that she measured \( |2^+_A\rangle \)

and Bob applies \( \sigma^B_x \)

\[ \sigma^B_x (\alpha |11\rangle_B + \beta |10\rangle_B ) = \alpha |10\rangle_B + \beta |11\rangle_B \]

State has been transferred from \( A \) to \( B \) without ever looking at it! Quantum teleportation.
Collapse of state is simultaneous
⇒ information transfer faster than light?

no! Without information which state Alice measured
Bob has no information encoded in his state since

\[ S_B = \text{Tr}_A \left\{ S_0 \right\} = 1 \]

**teleportation**

entangled state as resource
+ LOCC operation

### 2.4 Criterion and quantitative measure of entanglement of pure bi-partite states

as we have seen \( 12\psi \rangle \) is entangled if it cannot be written as

\[ 12\psi \rangle = |\psi \rangle_A |\phi \rangle_B \]

i.e. necessary and sufficient condition for entanglement is that Schmidt number \( S > 1 \).

for separable state

\[ S_A = \text{Tr}_B \left\{ 1 |2\psi \rangle \langle 2\psi| 1 \right\} = |\psi \rangle_A \langle \psi|, \text{ i.e. } S_A^2 = S_A \]

\[ S_B = \text{Tr}_A \left\{ 1 |2\psi \rangle \langle 2\psi| 1 \right\} = |\phi \rangle_B \langle \phi|, \text{ i.e. } S_B^2 = S_B \]

i.e. reduced density operator of each part is pure
Quantitative measure of entanglement?

Should have the following properties:

1. $E \geq 0$
2. $E = 0$ for Schmidt number $S = 1$
3. $E$ should be extensive (in sense of thermodynamics)
4. $E = \max$ if all coefficients of Schmidt decomposition equal

\[ E(1\psi) = - \text{Tr}_B \left\{ S_B \ln S_B \right\} = - \text{Tr}_A \left\{ S_A \ln S_A \right\} \]

i.e. von-Neumann entropy of subsystem

Remark: Schmidt decomposition $\dim(\mathcal{H}_A) = M \leq \dim(\mathcal{H}_B) = N$

\[ |\psi\rangle = \sum_{j=1}^{N} c_j |i\rangle_A |j\rangle_B \]

\[ S_B = \sum_{j=1}^{M} c_j^2 \ln \langle i|j \rangle_{BB} \]

\[ S_A = \sum_{j=1}^{M} c_j^2 \ln \langle i|j \rangle_{AA} \]

\[ E(1\psi) = - \text{Tr}_B \left\{ \sum_{j=1}^{N} c_j^2 |i\rangle\langle j| \ln \left( \sum_{k=1}^{N} c_k^2 |k\rangle\langle k| \right) \right\} \]
\( E = -\sum_{j=0}^{M} C_j \ln C_j \) \( \text{fulfills } (ii)-(iv) \)

\[ E_{\text{max}} = \ln M \quad \text{maximum entropy} \quad \ln \left( \min \left( \frac{1}{M}, \frac{1}{M} \right) \right) \]

One easily recognizes

A separable state \( |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \) cannot be transferred to an entangled pure state by LOCC

2.5 Entanglement of mixed bi-partite states

A pure state

\[ \rho = \left( |\psi_A\rangle \langle \psi_A| \langle \psi_B| \right) \quad \text{is separable} \]

We will still call \( \rho \) separable if it is a convex sum of separable pure states, with probabilities \( p_j \):

\[ \rho = \sum_{j=0}^{k} p_j \left( |\psi_A\rangle \langle \psi_A| \langle \psi_B| \right) \langle \psi_A| \langle \psi_B| \Rightarrow \text{separable}! \]

Otherwise \( \rho \) is called entangled. States of the form (27) can be generated by LOCC operations from pure separable states.

LOCC operations cannot turn separable states into entangled states.
A necessary condition for separability is given by a theorem of Peres. PRL 77, 1413 (1996).

\[ (28) \quad \text{if } \mathcal{S} \text{ is separable } \implies \mathcal{S}^{T_A} \geq 0 \text{ and } \mathcal{S}^{T_B} = (\mathcal{S}^{T_A})^T \geq 0 \]

\( \mathcal{S}^{T_A} \) is the partial transpose with respect to sub-system A.

**Proof:** \( \mathcal{S} \) separable
\[ \mathcal{S} = \sum_{j=1}^{K} p_j \left( |i_A\rangle\langle j_A| \otimes |i_B\rangle\langle j_B| \right) \]

\[ \mathcal{S}^{T_A} = \sum_{j=1}^{K} p_j \left( |i_A\rangle\langle j_A| \otimes |i_B\rangle\langle j_B| \right)^T \]

Using \( A^{\dagger} = (A^T)^* \), \( (|i\rangle\langle i|)^{\dagger} = |i\rangle\langle i| \)

\[ \implies \mathcal{S}^{T_A} = \sum_{j=1}^{K} p_j \left( |i_A^*\rangle\langle j_B| \otimes |i_B^*\rangle\langle j_A| \right) \]

\[ = \sum_{j=1}^{K} p_j \left( |i_A\rangle\langle j_B| \otimes |i_B\rangle\langle j_A| \right) \geq 0 \quad \square \]

H. P. and R. Horodecki have shown for \( \mathcal{H}_A = \mathbb{C}^2 \) and \( \mathcal{H}_B = \mathbb{C}^2 \) or \( \mathcal{H}_B = \mathbb{C}^3 \) Peres criterion is not only necessary but also sufficient.

**Horodecki Theorem**

\[ (29) \quad \text{in } \mathbb{C}^2 \otimes \mathbb{C}^2 \text{ or } \mathbb{C}^2 \otimes \mathbb{C}^3 \]

\( \mathcal{S} \) separable \( \iff \mathcal{S}^{T_A} \geq 0 \)
rem: a general practical necessary and sufficient condition for mixed-state entanglement is still not found

2.6 General entanglement measure:
entanglement of destruction, entanglement of formation

First approach:

Suppose Alice and Bob have $n$ copies of the same mixed bi-partite state $\rho_{AB}$. Suppose further they can generate out of these $k \leq n$ pure Bell states by LOCC operations. This corresponds to a destruction of a smaller number of maximally entangled pure states.

\[
E_D(\rho_{AB}) = \lim_{n \to \infty} \frac{\ln k_{\text{max}}}{n}
\]

Second approach:

Suppose Alice and Bob have an arbitrary number of Bell states at their disposal. Suppose further they want to create $n$ copies of $\rho_{AB}$ by LOCC operations using $k'$ Bell pairs. This corresponds to the creation of $\rho_{AB}$ out of maximally entangled pure states.
\[(31) \quad \mathcal{E}_f(S_{AB}) = \lim_{n \to \infty} \frac{\mathcal{H}_{\min}^n}{n} \]  

**entropy of formation**

(30) and (31) are good definitions but practically useless... For pure states one can show

\[(32) \quad \mathcal{E}_D(\mathcal{D}_1 \mathcal{D}_2) = \mathcal{E}_f(\mathcal{D}_1 \mathcal{D}_2) = \mathcal{E}(\mathcal{D}_1 \mathcal{D}_2)\]

where \(\mathcal{E}\) is the von-Neumann entropy.

For mixed states \(\mathcal{E}\) is no longer a good measure since also mixed states of the form of eq. (32) have \(\mathcal{E} > 0\).

**Relative entropy of entanglement \(\mathcal{E}_R\)**

\[(33) \quad \mathcal{E}_R(S_{AB}) = \min_{\mathcal{D} \in \text{ separable}} \text{Tr} \left\{ \mathcal{S} \mathcal{D} - \mathcal{S} \mathcal{D} \mathcal{S} \right\} \]

One can show for mixed states

\[(34) \quad \mathcal{E}_D(S_{AB}) \leq \mathcal{E}_R(S_{AB}) \leq \mathcal{E}_f(S_{AB}) \]

**Remark:** None of these measures are easily handleable
Concurrence

For a mixed state of two spin $1/2$ systems the concurrence is a useful entanglement measure:

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

where $\lambda_1, \ldots, \lambda_4$ are eigenvalues in decreasing order of

$$R = \left( \frac{1}{\sqrt{2}} \sqrt{\rho \rho^*} \right)^{1/2}$$

Logarithmic negativity

An upper bound to $E_D(\rho)$ can be found from

$$E_N(\rho) = \log_2 \| \rho^{T_A} \|$$

$\rho^{T_A}$ partial transpose and $\| x \| = \text{Tr} \sqrt{x^* x}$

Bound entanglement

Mixed states that are non-separable, however, with $E_D(\rho) = 0$ are not distillable and are called bound entangled. Since they are not distillable they cannot be used as an entanglement resource.
Horodecki's (PRL 80, 5239 (1998), Dür et al. (2000)

\[ \mathcal{E}_{\text{Tr}} \geq 0 \implies \mathcal{E} \text{ is non-desilliable} \]

i.e. it is either separable or at best bound entangled

for \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) also the inverse is true

\[ \mathbb{C}^2 \otimes \mathbb{C}^2 \]

\[ \text{general} \]

separable

= PPT

= non-desilliable

entangled

bound entangled

PPT: positive partial transpose