
Task 1.

The $W^{\pm}$ and Z fields of the weak interaction are massive vector fields. These can be described by the so-called Proca Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu. \tag{1}$$

(a) Derive the field equations for the A^μ.

(b) Determine the energy-momentum relation $E = E(\mathbf{p})$ for free Proca fields and explain with this the interpretation of m as mass.

(c) Show that (1) is not gauge invariant as long as $m \neq 0$.That is, gauge invariance and finite mass are mutually exclusive.

Task 2.

The Lagrangian density of the Schrödinger field is

$$\mathcal{L} = -\frac{i}{2} \left(\partial_t \Psi^*\right) \Psi + \frac{i}{2} \Psi^* \left(\partial_t \Psi\right) - \frac{1}{2m} \nabla \Psi^* \nabla \Psi. \tag{2}$$

(a) Show that (2) leads to the Schrödinger equation. Note that Ψ is complex-valued.

(b) Calculate the generalized momenta π.

(c) What is the Hamiltonian H?

Task 3.

Every vector $\mathbf{F}(\mathbf{r})$ can be decomposed into a longitudinal part $\mathbf{F}_\parallel(\mathbf{r})$ and a transverse part $\mathbf{F}_\perp(\mathbf{r})$, where $\nabla \cdot \mathbf{F}_\perp = 0$ and $\nabla \times \mathbf{F}_\parallel = 0$.

(a) Show that

$$\mathbf{F}_\perp(\mathbf{x}) = \int \mathrm{d}^3\mathbf{y} \; \overset{\leftrightarrow}{\delta}_\perp (\mathbf{x} - \mathbf{y}) \, \mathbf{F}(\mathbf{y}), \tag{3}$$

$$\mathbf{F}_\parallel(\mathbf{x}) = \int \mathrm{d}^3\mathbf{y} \; \overset{\leftrightarrow}{\delta}_\parallel (\mathbf{x} - \mathbf{y}) \, \mathbf{F}(\mathbf{y}), \tag{4}$$

where

$$\overleftrightarrow{\delta}_{\parallel}(\mathbf{x}) = -\nabla \otimes \nabla \frac{1}{4\pi|\mathbf{x}|}, \tag{5}$$

and

$$\overleftrightarrow{\delta}_{\perp}(\mathbf{x}) = \delta(\mathbf{x}) - \overleftrightarrow{\delta}_{\parallel}(\mathbf{x}). \tag{6}$$

(b) Let $\boldsymbol{\phi}(\mathbf{x})$ and $\boldsymbol{\pi}(\mathbf{x})$ be canonically conjugate vector fields. Show that

$$\left\{\boldsymbol{\phi}_{\perp}(\mathbf{x}), \boldsymbol{\pi}_{\perp}(\mathbf{y})\right\} = \overleftrightarrow{\delta}_{\perp}(\mathbf{x}-\mathbf{y}). \tag{7}$$