
Task 18.

The Hamiltonian of the relativistic electron-positron field interacting with the quantized radiation field in an external central potential $V(r)$ is given by

$$\hat{H} = \int \mathrm{d}^3\mathbf{r}\ \hat{\Psi}^\dagger \left[\boldsymbol{\alpha}\cdot\left(\frac{1}{i}\nabla - e\hat{\mathbf{A}}\right) + \beta m + V(r)\right]\hat{\Psi} + \hat{H}_{elm}.$$

Show that this transforms in non-relativistic limit to

$$\hat{H} = \int \mathrm{d}^3\mathbf{r}\ \hat{\Phi}^\dagger \left[\frac{1}{2m}\left(\frac{1}{i}\nabla - e\hat{\mathbf{A}}\right)^2 - \frac{e}{m}\hat{\mathbf{S}}\cdot\hat{\mathbf{B}} + V(r)\right]\hat{\Phi} + \hat{H}_{elm}, \qquad (1)$$

wher $\hat{\mathbf{S}} = \frac{1}{2}\hat{\boldsymbol{\sigma}}$ is the spin operator and $\hat{\Phi}$ is a second-component spinor field.

Task 19.

Neglecting the spin in Eq. (1), yields the non-relativistic Hamiltonian

$$\begin{aligned}
\hat{H}_0 &= \hat{H}_{elm} + \int \mathrm{d}^3\mathbf{r}\ \hat{\Psi}^\dagger \left[-\frac{1}{2m}\nabla^2 + V(r)\right]\hat{\Psi}, \\
\hat{H}_I &= \int \mathrm{d}^3\mathbf{r}\ \hat{\Psi}^\dagger \left[-\frac{i}{m}\nabla\cdot\hat{\mathbf{A}} + \frac{e^2}{2m}\hat{\mathbf{A}}^2\right]\hat{\Psi}, \\
\hat{H} &= \hat{H}_0 + \hat{H}_I.
\end{aligned}$$

Let $|\alpha\rangle = |a\rangle|0\rangle$ and $|\beta\rangle = |b\rangle|1_{\mathbf{k},\lambda}\rangle$ be eigenstates of $\hat{H}_0$. Show in lowest order perturbation theory that the transition probability per unit time for the spontaneous emission is given by:

$$\Delta W_{ba} = \sum_{\{1_{\mathbf{k},\lambda}\}} \Delta W_{\beta\alpha} = \frac{\omega_{ba}^3}{3\pi}\left|\mathbf{d}_{ba}\right|^2,$$

where $\mathbf{d}_{ba}$ is the dipole matrix element of the transition $|b\rangle \to |a\rangle$.

Task 20.

Let $f(x)$ be a real function which has a Fourier representation, i.e.

$$f(x) = \int \mathrm{d}k\widetilde{f}(k)\,\mathrm{e}^{ikx}.$$

Show that:

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \mathrm{d}x\,\frac{f(x)}{x_0 - x + i\epsilon} = -i\pi f(x_0) + \mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}x\,\frac{f(x)}{x_0 - x} \qquad (2)$$

(Hint: Use Cauchy's integral theorem for integration in complex plane) From (2) the following relation can be deduced:

$$\lim_{\epsilon\to 0}\frac{1}{x_0 - x + i\epsilon} = -i\pi\delta(x - x_0) + \mathcal{P}\frac{1}{x_0 - x}.$$