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**Task 4.** (Noether Theorem I)

Consider a field theory with 2 types of Dirac spinors

$$\mathcal{L} = \sum_{l=1}^{2} \overline{\Psi}^{(l)} \left[ \frac{i}{2} \gamma^\mu \overleftrightarrow{\partial}_\mu - m \right] \Psi^{(l)}.$$

Global phase transformations are performed by the elements of the $SU(2)$

$$U = \exp\left(-ig\sigma^j \theta_j\right)$$

where, $\sigma^j$ are Pauli matrices and $\theta_j$ are independent continuous parameters.
Show that according to Noether's Theorem there are 3 conserved currents of the form

$$j_i^\mu = g\overline{\Psi}\gamma^\mu \sigma_i \Psi, \qquad i \in \{1, 2, 3\}.$$

**Task 5.** (Noether Theorem II)

Show that the invariance of a Lagrangian density $\mathcal{L}$ under spatial and temporal translations satisfies the conservation equation

$$\partial_\mu T^{\mu\nu} \;=\; 0 \tag{1}$$

with

$$T^{\mu\nu} \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x_\nu} - g^{\mu\nu} \mathcal{L} \tag{2}$$

Determine the four conserved Noether charges

$$Q^\nu \;\equiv\; \int \mathrm{d}^3\mathbf{x}\, T^{0\nu} \tag{3}$$

for the special case of the electromagnetic field.

**Task 6.** (Free Dirac equation)

Determine the stationary solutions of the free Dirac equation ($\hbar = c = 1$)

$$i\partial_t \Psi = \left(-i\overrightarrow{\alpha} \cdot \overrightarrow{\nabla} + m\beta\right) \Psi$$

with the ansatz

$$\Psi = \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} \exp\left(i\overrightarrow{p} \cdot \overrightarrow{r} - iEt\right).$$

**Task 7.** (non-relativistic limit and Dirac-Pauli equation)

Consider the Dirac equationg

$$i\partial_t \Psi = \left(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta\right)\Psi$$

for the bispinor $\Psi$. Show that in the non-relativistic limit using the ansatz

$$\Psi\left(\vec{r},t\right) = \begin{pmatrix} \varphi\left(\vec{r},t\right) \\ \chi\left(\vec{r},t\right) \end{pmatrix} \exp\left(-imt\right)$$

where $\phi$ and $\chi$ are two-component objects, follows

$$\chi\left(\vec{r},t\right) \simeq \frac{\vec{\sigma}\cdot\left(-i\vec{\nabla}\right)}{2m}\varphi\left(\vec{r},t\right).$$

i.e. $|\chi| \ll |\phi|$, and

$$i\partial_t\varphi = -\frac{\vec{\nabla}^2}{2m}\varphi.$$

This is precisely the Schrödinger equation for a two-component wave function (Pauli equation).