
Task 15. Covariant Heisenberg equations of motion

From the translation invariance of the Lagrangian density follows the existence of a conserved and time-independent four-vector

$$P^\mu = \int \mathrm{d}^3\mathbf{x} \; : \pi^r(x)\partial^\mu \phi^r(x) - g^{\mu 0}\mathcal{L} : .$$

Here $\phi^r(x)$ are real-valued fields and π_r the corresponding canonical momenta and r denotes the number of field components. Verify the validity of the covariant Heisenberg equation of motion

$$[\phi_r(x), P^\mu] = i\partial^\mu \phi_r(x) \tag{1}$$

$$[\pi_r(x), P^\mu] = i\partial^\mu \pi_r(x). \tag{2}$$

(Note: Here $x = (t, \mathbf{x})^T$ and $\mathbf{x} = (x_1, x_2, x_3)^T$ and $: \cdots :$ means normal order*.)

* Normal ordering of a function of annihilation and creation operators means to move all creation operators to the left and all annihilation operators the right.

Task 16. Lorentz invariance of the mode decomposition of scalar fields

If one wants to understand how Lorentz transformations act on the mode operators $\hat{a}_\mathbf{k}$, $\hat{a}^\dagger_\mathbf{k}$ of a scalar field $\hat{\phi}(x)$, one cannot work with quantization in a box, as it is not Lorentz-invariant. Therefore, one considers $V \to \infty$ and replaces the discrete operators $\hat{a}_\mathbf{k}$ with continuous operators $\hat{a}(\mathbf{k})$ by means of

$$\hat{a}(\mathbf{k}) = \hat{a}_\mathbf{k}\sqrt{2V\omega_\mathbf{k}}.$$

The decomposition of the field into normal modes can then be written as

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2\omega_\mathbf{k}} \left[\hat{a}(\mathbf{k})\mathrm{e}^{-ikx} + \hat{a}^\dagger(\mathbf{k})\mathrm{e}^{ikx}\right].$$

Show that these substitutions lead to the following commutation relations:

$$\left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')\right] = (2\pi)^3 2\omega_\mathbf{k}\delta(\mathbf{k} - \mathbf{k}').$$

Task 17.

Let $\hat{a}$, $\hat{a}^\dagger$ be bosonic annihilation and creation operators. Show:

$$\hat{a} f(\hat{n}) = f(\hat{n}+1)\hat{a}, \tag{3}$$

$$\hat{a}^\dagger f(\hat{n}) = f(\hat{n}-1)\hat{a}^\dagger, \tag{4}$$

$$\left[\hat{a}, \hat{a}^{\dagger l}\right] = l\hat{a}^{\dagger(l-1)} = \frac{\partial \hat{a}^{\dagger l}}{\partial \hat{a}^\dagger}, \tag{5}$$

$$\left[\hat{a}^\dagger, \hat{a}^l\right] = -l\hat{a}^{(l-1)} = -\frac{\partial \hat{a}^l}{\partial \hat{a}}, \tag{6}$$

$$\left[\hat{a}, f(\hat{a}^\dagger)\right] = \frac{\partial f(\hat{a}^\dagger)}{\partial \hat{a}^\dagger}, \tag{7}$$

$$\left[\hat{a}^\dagger, f(\hat{a})\right] = -\frac{\partial f(\hat{a})}{\partial \hat{a}}. \tag{8}$$

Using the general relationship for two operators $\hat{A}$, $\hat{B}$

$$\mathrm{e}^{\xi\hat{A}}F\left(\hat{B}\right)\mathrm{e}^{-\xi\hat{A}} = F\left(\mathrm{e}^{\xi\hat{A}}\hat{B}\mathrm{e}^{-\xi\hat{A}}\right)$$

show that

$$\mathrm{e}^{x\hat{a}}f\left(\hat{a},\hat{a}^{\dagger}\right)\mathrm{e}^{-x\hat{a}} = f\left(\hat{a},\hat{a}^{\dagger}+x\right), \quad (9)$$

$$\mathrm{e}^{-x\hat{a}^{\dagger}}f\left(\hat{a},\hat{a}^{\dagger}\right)\mathrm{e}^{x\hat{a}^{\dagger}} = f\left(\hat{a}+x,\hat{a}^{\dagger}\right) \quad (10)$$

and also

$$\mathrm{e}^{x\hat{a}^{\dagger}\hat{a}}f\left(\hat{a},\hat{a}^{\dagger}\right)\mathrm{e}^{-x\hat{a}^{\dagger}\hat{a}} = f\left(\hat{a}\mathrm{e}^{-x},\hat{a}^{\dagger}\mathrm{e}^{x}\right).$$