4. Exercise

Task 12.

The Hamiltonian and the momentum of the electromagnetic field in Coulomb gauge are given by:

$$\hat{H} = \frac{1}{2}\int \mathrm{dV}\left(\hat{\mathbf{E}}_\perp^2 + \hat{\mathbf{B}}_\perp^2\right), \tag{1}$$

$$\hat{\mathbf{P}} = \int \mathrm{dV}\left(\hat{\mathbf{E}}_\perp \times \hat{\mathbf{B}}_\perp\right). \tag{2}$$

Show that

$$\hat{\mathbf{P}} = \sum_{\mathbf{k},\lambda} \mathbf{k}\hat{a}_\lambda^\dagger(\mathbf{k})\hat{a}_\lambda(\mathbf{k}), \tag{3}$$

$$\omega_{\mathbf{k}}\hat{a}_\lambda(\mathbf{k}) = \left[\hat{a}_\lambda(\mathbf{k}), \hat{H}\right], \tag{4}$$

$$\mathbf{k}\hat{a}_\lambda(\mathbf{k}) = \left[\hat{a}_\lambda^\dagger(\mathbf{k}), \hat{\mathbf{P}}\right]. \tag{5}$$

Task 13. Sum rule of the Maxwell field

Show that

$$\sum_{k=1}^{3}\left[\hat{A}_\perp^k(\mathbf{x},t), \hat{E}_\perp^k(\mathbf{y},t)\right] = -2i\delta(\mathbf{x}-\mathbf{y}).$$

Task 14. Angular momentum of the electromagnetic field

Show that the total angular momentum operator of the electromagnetic field

$$\hat{\Omega}^j = i\int \mathrm{d}^3\mathbf{r}\ \hat{E}_\perp^k\left(L^j\hat{A}^k - i\epsilon_{jkl}\hat{A}^l\right)$$

where

$$\mathbf{L} = -i\mathbf{r}\times\nabla$$

is a generator for infinitesimal rotations of the Maxwell field.