Task 8. Bose/Fermi commutation rules

Show that the commutation rules of the single-particle operators

$$\hat{\mathbf{p}} = \int \mathrm{d}^3\mathbf{x}\, \hat{\psi}^\dagger(x) \left(-\frac{\hbar}{i}\nabla\right) \hat{\psi}(x)$$

and

$$\hat{\mathbf{x}} = \int \mathrm{d}^3\mathbf{x}\, \hat{\psi}^\dagger(x)\mathbf{x}\hat{\psi}(x)$$

apply independently of whether Bose or Fermi commutation rules are valid for $\hat{\psi}$.

Task 9. Quantization of the electromagnetic field

In Coulomb gauge, the operator of the transverse vector potential in the volume L^3 with periodic boundary conditions can be decomposed into normal modes as follows:

$$\hat{\mathbf{A}}_\perp(\mathbf{r},t) = \sum_n \sum_{\alpha=1}^{2} \sqrt{\frac{\hbar}{2c|k_n|L^3}} \left(\mathbf{e}_{n\alpha} \mathrm{e}^{i\mathbf{k}_n\mathbf{r}-i\omega_n t}\hat{a}_{n\alpha} + \text{h.a.}\right), \tag{1}$$

where $\mathbf{k}_n = (k_{n_1}, k_{n_2}, k_{n_3})$, $k_{n_i} = \frac{2\pi n_i}{L}$, $n_i = 0, \pm1, \pm2, ...$ and $\omega_n = c|\mathbf{k}_n|$.
Show that the commutation relations of the creation and annihilation operators

$$[\hat{a}_{n\alpha}, \hat{a}_{m\beta}] = 0 \qquad \left[\hat{a}_{n\alpha}, \hat{a}^\dagger_{m\beta}\right] = \delta_{nm}\delta_{\alpha\beta} \tag{2}$$

are equivalent to the canonical commutation relations

$$\left[\hat{A}^k_\perp(\mathbf{r},t), \hat{E}^l_\perp(\mathbf{r}',t)\right] = -i\hbar\delta^{kl}_\perp(\mathbf{r}-\mathbf{r}'). \tag{3}$$

Task 10.

The Hamiltonian operator of the Hermitian Klein-Gordon fields is $\left(\dot{\hat{\phi}} = \hat{\pi}\right)$

$$\hat{H} = \frac{1}{2}\int \mathrm{dV} \left(\dot{\hat{\phi}}^2 + \left(\nabla\hat{\phi}\right)^2 + m^2\hat{\phi}^2\right).$$

Show that the following commutation relations hold

$$\left[\hat{\phi}(x), \hat{H}\right] = i\hat{\pi}(x), \tag{4}$$

$$\left[\hat{\pi}(x), \hat{H}\right] = -i\left(m^2 - \nabla^2\right)\hat{\phi}(x). \tag{5}$$

In what way do these equations imply the Klein-Gordon equation for $\hat{\phi}$?

Task 11.

Consider the quantized Schrödinger field with 2-particle interaction, i.e.,

$$\hat{H} = \int d^3\overrightarrow{r}\hat{\Psi}^\dagger\left(\overrightarrow{r}\right)\left[-\frac{1}{2m}\Delta\right]\hat{\Psi}\left(\overrightarrow{r}\right) + \int d^3\overrightarrow{r}\int d^3\overrightarrow{r}'\hat{\Psi}^\dagger\left(\overrightarrow{r}\right)\hat{\Psi}^\dagger\left(\overrightarrow{r}'\right)V\left(\overrightarrow{r}-\overrightarrow{r}'\right)\hat{\Psi}\left(\overrightarrow{r}\right)\hat{\Psi}\left(\overrightarrow{r}'\right).$$

Derive equations of motion for the one- and two-particle wave functions using the Heisenberg equations for the field $\hat{\Psi}$

$$\begin{aligned}\Phi\left(\overrightarrow{r},t\right) &= \langle 0|\,\hat{\Psi}\left(\overrightarrow{r},t\right)|\varphi\rangle \\ \Phi\left(\overrightarrow{r}_1,\overrightarrow{r}_2,t\right) &= \langle 0|\,\hat{\Psi}\left(\overrightarrow{r}_1,t\right)\hat{\Psi}\left(\overrightarrow{r}_2,t\right)|\varphi\rangle\,.\end{aligned}$$