VII. Some remarks about the standard model

VII.1. Gauge theories for fermions with more degrees of freedom: non-Abelian gauge theories

An important guide to "construct" QED was the requirement of local gauge invariance which has fixed the minimal coupling term between the Dirac and a bosonic gauge field, the elm. field

(548) $$\psi(x) \longrightarrow e^{-i\theta(x)}\psi(x)$$

$e^{i\theta(x)}$ is an element of the Abelian $U(1)$ Lie group. Invariance under this trafo required the existence of a gauge field $A_\mu(x)$ coupled to ψ through the replacement

(549) $$\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu + ieA_\mu(x)$$

This resulted in

(550) $$\mathcal{L}_D = \frac{1}{2}\bar{\psi}(i\gamma^\mu \overleftrightarrow{\partial}_\mu - m)\psi \rightarrow \mathcal{L} = \mathcal{L}' - j^\mu A_\mu$$

The only bilinear and Lorentz-invariant construction of a Lagrangian of the gauge field are

$$(551)\quad \mathcal{L}_A = \lambda_1 F_{\mu\nu} F^{\mu\nu} + \lambda_2 G_{\mu\nu} G^{\mu\nu} + m_\gamma^2 A^\mu A_\mu$$

where $\quad F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$

$$G_{\mu\nu} = A_{\mu,\nu} + A_{\nu,\mu}$$

The only gauge invariant term is however

$$\lambda_1 F_{\mu\nu} F^{\mu\nu} \qquad (A_\mu \to A_\mu + \partial_\mu \Lambda)$$

i.e.

$$(552)\qquad \lambda_2 \equiv 0 \qquad m_\gamma \equiv 0$$

which than fixes the total Lagrangian to

$$(553)\quad \mathcal{L} = \frac{1}{2} \bar{\psi} (i\gamma^\mu \overleftrightarrow{\partial}_\mu - m) \psi - j^\mu A_\mu + \lambda F_{\mu\nu} F^{\mu\nu}$$

It turns out that there are other fermionic elementary particles which are described by additional degrees of freedom

Weak interaction

Isospin $\longrightarrow$ doublett of fermions

Strong interaction

Color $\longrightarrow$ triplett of fermions

In view of the success of the principle of local gauge invariance in QED we here want to request local gauge invariance too! So what are the allowed gauge transformations for two- or three-component fermions?

Weak interaction

(554) $$\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_1'(x) \\ \psi_2'(x) \end{pmatrix} = \underline{\underline{U}}(x) \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

where ψ_1, ψ_2 are Dirac bispinors. $\underline{\underline{U}}(x)$ is now a unitary 2×2 matrix in the isospin degrees

(555) $$\underline{\underline{U}}(x) = e^{-i\frac{1}{2}\tau_e \theta_e(x)}$$

$\tau_e \in (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices and $\theta_x, \theta_y, \theta_z$ are three independent angles

(556) $$\underline{\underline{U}}(x) \in SU(2)$$

which is a non-Abelian Lie group with generators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Strong interaction

$$(557)\quad \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_1'(x) \\ \psi_2'(x) \\ \psi_3'(x) \end{pmatrix} = \underline{U}(x) \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}$$

$$(558)\quad \text{where} \qquad \underline{\underline{U}}(x) \in SU(3)$$

non-Abelian Lie group

is a 3×3 matrix in the color degrees.
One can show that

$$(559)\quad \underline{\underline{U}}(x) = e^{-i\frac{1}{2}\sum_{k=1}^{8} \underline{\underline{\lambda}}_k \Theta_k(x)}$$

where $\underline{\underline{\lambda}}_1, \underline{\underline{\lambda}}_2, \dots, \underline{\underline{\lambda}}_8$ are the 8

Gell-Mann matrices

$$\underline{\underline{\lambda}}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{\underline{\lambda}}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{\underline{\lambda}}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{\lambda}}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \underline{\underline{\lambda}}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \underline{\underline{\lambda}}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(560)\quad \underline{\underline{\lambda}}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \underline{\underline{\lambda}}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$SU(n)$ has n^2-1 generators

$$(561)\quad [\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c$$

f_{abc} are the structure factors, which define a Lie algebra. Furthermore

(562) $$Tr\{\lambda_a \lambda_b\} = 2\delta_{ab}$$

Non-Abelian gauge theory

For a theory to be invariant under a $SU(n)$ transformation

$$\underline{\psi} = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix} \longrightarrow \underline{\underline{U}}(x) \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix} = e^{-i\frac{1}{2}\underline{\underline{\lambda_k}}\theta_k(x)} \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix}$$

the required gauge field is an $n \times n$ matrix

(563) $$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \underline{\bar{\psi}}\, i\gamma^\mu \left(\underline{\underline{U}}^+ \partial_\mu \underline{\underline{U}}\right) \underline{\psi}$$

so

(564) $$\boxed{\vec{\partial}_\mu \longrightarrow \vec{\underline{D}}_\mu = \vec{\partial}_\mu + ig\,\underline{\underline{A}}_\mu} \qquad \underline{\underline{A}}_\mu \quad n \times n$$

decomposing $\underline{\underline{A}}_\mu$ into the complete set of (n^2-1) $\underline{\lambda}_k$ gives n^2-1 gauge fields A_μ^a $a = 1, 2, \dots, n^2-1$

(565) $$\boxed{\underline{\underline{A}}_\mu(x) = \frac{1}{2}\sum_{a=1}^{n^2-1} \underline{\underline{\lambda}}_a A_\mu^a(x)}$$

The new interaction Lagrangian then reads

(566) $$\boxed{\mathcal{L}_{int} = -g\, \underline{\bar{\Psi}}\, \gamma^\mu \underline{\underline{A}}_\mu(x)\, \underline{\Psi}}$$

Using (565) one recognizes that the theory has now (n^2-1) Noether currents instead of just one as QED

(567) $$\boxed{j^\mu_e(x) = g\, \underline{\bar{\Psi}}(x)\, \gamma^\mu \underline{\underline{\lambda}}_e\, \underline{\Psi}(x)}$$

$e = 1, 2, \dots, n^2-1$

(Note that the γ^μ act on the bispinor degrees of all of the n components of $\underline{\Psi}(x)$.)

How about the free Lagrangian of the n^2-1 gauge fields?

$$\underline{\underline{F}}_{\mu\nu} = \partial_\mu \underline{\underline{A}}_\nu - \partial_\nu \underline{\underline{A}}_\mu$$ ↯ not gauge invariant

In contrast to $U(1)$ where definition of $F_{\mu\nu}$ with ∂_μ and D_μ are the same, one here has to define

(568) $$\boxed{\underline{\underline{F}}_{\mu\nu} = D_\mu \underline{\underline{A}}_\nu - D_\nu \underline{\underline{A}}_\mu}$$

i.e.

$$\underline{\underline{F}}_{\mu\nu} = \partial_\mu \underline{\underline{A}}_\nu + iq \underline{\underline{A}}_\mu \underline{\underline{A}}_\nu - \partial_\nu \underline{\underline{A}}_\mu - iq \underline{\underline{A}}_\nu \underline{\underline{A}}_\mu$$

$$\boxed{\underline{\underline{F}}_{\mu\nu} = \partial_\mu \underline{\underline{A}}_\nu - \partial_\nu \underline{\underline{A}}_\mu + iq \underbrace{[\underline{\underline{A}}_\mu, \underline{\underline{A}}_\nu]}_{\text{in general } \neq 0}} \tag{569}$$

$\Rightarrow$ field tensor of $SU(n)$ is nonlinear in $\underline{\underline{A}}_\mu(x)$!

The gauge-invariant Lagrangian of free gauge fields is

$$\boxed{\mathcal{L}_{gauge} = -\frac{1}{2} \mathrm{Tr}\{\underline{\underline{F}}_{\mu\nu} \underline{\underline{F}}^{\mu\nu}\}} \tag{570}$$

Here "Tr" is over the n internal degrees (isospin, color...)

Due to the commutator in (569) the free Lagrangian is no longer quadratic in the fields

$\Rightarrow$ intrinsic self-interaction of gauge fields

The complete Lagrangian of a $SU(N)$ local gauge-invariant field theory, called

Yang-Mills theory

is

$$\boxed{\mathcal{L} = \overline{\Psi}(x)\left[\frac{i}{2}\gamma^\mu \overleftrightarrow{\underline{\underline{D}}}_\mu - m\right]\underline{\Psi}(x) - \frac{1}{2}\mathrm{Tr}\left\{\underline{\underline{F}}_{\mu\nu}\underline{\underline{F}}^{\mu\nu}\right\}}$$

(577)

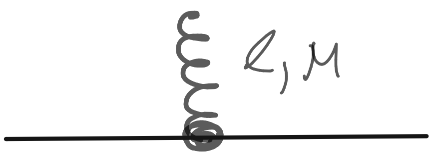
there are different interaction terms

ffg coupling

in SU(N) each fermion ("f") has N internal degrees of freedom and couples to (N^2-1) gauge bosons ("g")

(572) $$\boxed{\mathcal{L}_{ffg} = -g\,\overline{\underline{\Psi}}\,\gamma^\mu \frac{1}{2}\underline{\underline{\lambda}}_\ell\,\underline{\Psi}\,A^\ell_\mu} \qquad \ell = 1,2,\dots N^2-1$$

vertex



$$-ig\gamma^\mu \frac{1}{2}\underline{\underline{\lambda}}_\ell$$

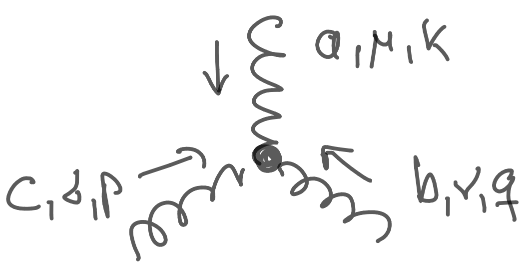
here $\underline{\underline{\lambda}}_\ell$ mixes the internal degrees of fermion

The interaction contained in the free Lagrangian of the gauge field are

(573)
$$\mathcal{L}_F = -\frac{1}{2}\partial^\mu A^{e\nu}\left(\partial_\mu A^e_\nu - \partial_\nu A^e_\mu\right) + \mathcal{L}_{3g} + \mathcal{L}_{4g}$$

3g - coupling

(574)
$$\boxed{\mathcal{L}_{3g} = \frac{1}{2} g f_{abc} A^{b\mu} A^{c\nu}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)}$$

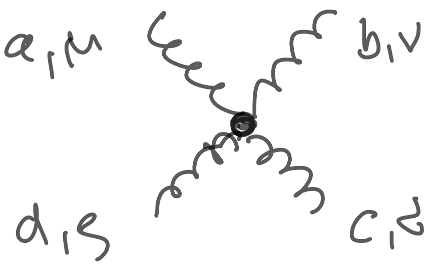


a, b, c isospin, color
μ, ν, σ Lorentz index (polarization)
k, p, q 4-momenta

vertex:
$$g f_{abc}\left\{ g_{\mu\nu}(q_\sigma - k_\sigma) + g_{\nu\sigma}(p_\mu - q_\mu) + g_{\sigma\mu}(k_\nu - p_\nu)\right\}$$

4g - coupling

(575)
$$\boxed{\mathcal{L}_{4g} = -\frac{1}{4} g^2 f_{abe} f_{cde} A^a_\mu A^b_\nu A^{d\mu} A^{c\nu}}$$



vertex:
$$-ig^2\left\{ f_{abe}f_{cde}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right) + f_{ace}f_{bde}\left(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\nu\sigma}\right) + f_{ade}f_{cbe}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}\right)\right\}$$

VII.2 Spontaneous symmetry breaking and Higgs mechanism

A fundamental problem of the Yang-Mills theory is the absence of masses of the gauge bosons required by gauge invariance. We will now discuss how these masses can be generated by the Higgs mechanism.

(A) Spontaneous symmetry breaking

Let us first discuss a phenomenon called spontaneous symmetry breaking.

idea: If a Hamiltonian is invariant under a certain symmetry group the ground state may not have this symmetry

conditions:
- invariance under symmetry group $[H, G] = 0$
- ground state of H is degenerate and transforms non-trivially under symmetry group

classical theory: scalar real field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi^2) \qquad (576)$$

invariant under

$$P: \quad \phi \to -\phi$$

- assume non-degenerate ground state $|0\rangle$

(577) $$P|0\rangle = e^{i\varphi}|0\rangle$$

$$\Rightarrow \langle 0|\hat{\phi}|0\rangle = \langle 0|P^{-1}P\hat{\phi}P^{-1}P|0\rangle$$

$$= \langle 0|P\hat{\phi}P^{-1}|0\rangle = -\langle 0|\hat{\phi}|0\rangle$$

$$\Rightarrow \underline{\underline{\langle 0|\hat{\phi}|0\rangle = 0}}$$

- but for degenerate ground state, e.g. for

(578) $$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda^2}{4}\phi^4$$

potential has two minima

(579) $$\phi_0 = \pm\sqrt{\frac{\mu^2}{\lambda^2}} = \pm v$$

two minimum energy solutions $\phi_0 = \pm v$

$|R\rangle, |L\rangle$

(580) $$P|R\rangle = |L\rangle \neq |R\rangle$$

$$\Rightarrow \langle R|\hat{\phi}|R\rangle = \langle R|\hat{P}^{-1}\underbrace{\hat{P}\,\hat{\phi}\,\hat{P}^{-1}}_{=-\hat{\phi}}\underbrace{\hat{P}\,|R\rangle}_{=|L\rangle} \tag{581}$$
$$= -\langle L|\hat{\phi}|L\rangle$$

i.e. expectation value of the field in ground state is non zero despite symmetry of $\mathcal{L}$

$\Rightarrow$ spontaneously broken discrete symmetry

More interesting is a spontaneously broken continuous symmetry

two scalar fields $\quad \vec{\phi} = (\phi_1, \phi_2)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\phi}\cdot\partial^\mu\vec{\phi} + \frac{1}{2}\mu^2\vec{\phi}\cdot\vec{\phi} - \frac{\lambda^2}{4}(\vec{\phi}\cdot\vec{\phi})^2 \tag{582}$$

$\mathcal{L}$ has $O(2)$ symmetry (rotation in ϕ_1, ϕ_2 plane)

$$R(\theta)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix}$$

leaves $\mathcal{L}$ invariant

$\mu^2 < 0$: unique ground state $\vec{\phi} = 0$

$\mu^2 > 0$: infinitely many degenerate ground states with

$$(583) \qquad |\vec{\phi}|^2 = \phi_1^2 + \phi_2^2 = v^2$$

If we pick one ground state as reference state $|0\rangle$ then

$$(584) \qquad R(\theta)|0\rangle = |\theta\rangle \quad \text{yields all others}$$

quantum theory

In the quantum case there is (except for pathological cases such as infinite potential barriers) strictly speaking no spontaneous symmetry breaking due to quantum tunneling.

However the tunneling time scales as

$$T = \frac{2\pi}{\Delta E} \quad \text{and} \quad \Delta E \sim \mathcal{O}(L^{-3})$$

i.e. in the thermodynamic limit $L^3 \to \infty$, $T \to \infty$

(B) Goldstone theorem

In a covariant and local field theory a spontaneously broken continuous symmetry leads to a massless excitation, called Goldstone boson, if Hilbert space norm is positive.

example from (582) with $\mu^2 > 0$

expand $V(\vec{\phi})$ around minimum, say $\vec{\phi}_0 = (0, v)$

(585) $$V(\vec{\phi}) \cong V(\vec{\phi}_0) + \frac{1}{2} \underbrace{\frac{\partial^2 V}{\partial \phi_e \partial \phi_m}\bigg|_{\vec{\phi}_0}}_{M^2_{em}} (\phi_e - v\delta_{e2})(\phi_m - v\delta_{m2})$$

Mass matrix M^2_{em}

$$M^2_{em} = \begin{pmatrix} 0 & 0 \\ 0 & 2\mu^2 \end{pmatrix} \qquad m \equiv 2\mu^2$$

Now let us define new field $(\phi_1, \underbrace{\phi_2 - v}_{=\chi})$

(586) $$V \cong \frac{m^4}{16\lambda^2} + \frac{1}{2} m^2 \chi^2 + \sqrt{\frac{m^2\lambda^2}{2}}\, \chi (\phi_1^2 + \chi^2) + \frac{\lambda^4}{4} (\phi_1^2 + \chi^2)^2$$

mass term for χ

There is no mass term for ϕ_1! ϕ_1 - massless

Goldstone holds if

(i) theory is explicitly covariant

and (ii) Hilbert-space norm is positive

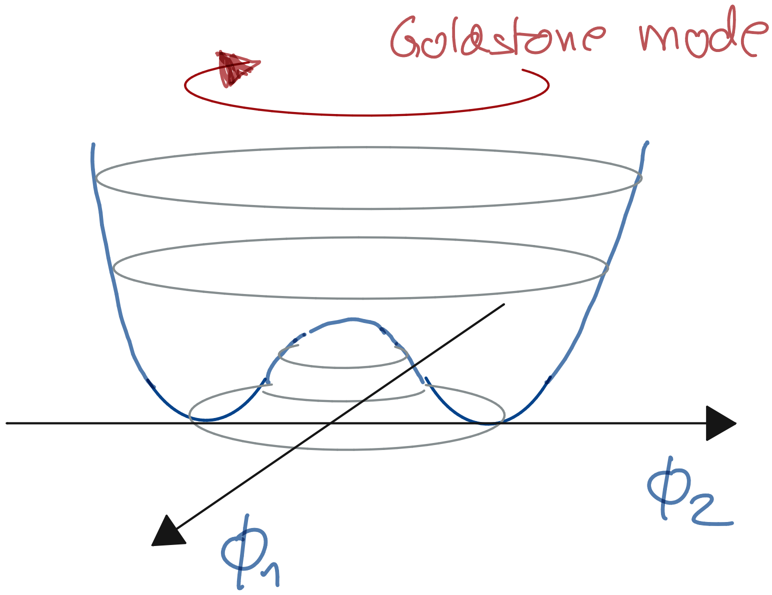
This is e.g. not the case for electromagnetism where one either has to choose a gauge where explicit covariance is no longer given (Coulomb gauge) or to work in bigger Hilbert space with indefinite norm (Lorentz gauge) where "physical" states are obtained with additional constraints (Gupta-Bleuler)

(c) Higgs mechanism

Consider again Lagrangian (582) $\vec{\phi} = (\phi_1, \phi_2)$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} + \frac{1}{2} \mu^2 \vec{\phi} \cdot \vec{\phi} - \frac{\lambda^2}{4} (\vec{\phi} \cdot \vec{\phi})^2$$

with $\phi_1^* = \phi_1, \phi_2^* = \phi_2$



polar coordinates

$$\rho = \sqrt{\phi_1^2 + \phi_2^2}$$
$$\sin\theta = \frac{\phi_2}{\sqrt{\phi_1^2 + \phi_2^2}} \qquad (587)$$

The Goldstone mode corresponds to the angular degree of freedom.

Now lets couple the fields to a gauge field. For this lets introduce complex variables

(588) $$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) = \frac{1}{\sqrt{2}} \rho e^{i\theta}$$

$\Rightarrow$

(589) $$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \mu^2 \phi^* \phi - \lambda^2 (\phi^* \phi)^2$$

The $O(2)$ transformation $R(\alpha)$ is now a phase transformation $\phi \to \phi e^{i\alpha}$

Minimum coupling to a gauge field A_μ:

(590) $$\partial_\mu \phi \longrightarrow (\partial_\mu + igA_\mu)\phi$$

Then

(591) $$\mathcal{L} \to \mathcal{L} = (\partial_\mu - igA_\mu)\phi^* (\partial^\mu + igA^\mu)\phi + \mu^2 \phi^* \phi - \lambda^2 (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Now we choose a special gauge: unitary gauge

(592) $$\phi \to \phi' = \phi e^{-i\theta(x)} = \phi'^*$$

(593) $$A_\mu \to A_\mu' = A_\mu + \frac{1}{g}\partial_\mu \theta(x)$$

this gives the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu - igA_\mu)\rho(\partial^\mu + igA^\mu)\rho + \frac{\mu^2}{2}\rho^2 - \frac{\lambda^4}{4}\rho^4$$

(594) $$- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$\mathcal{L}$ does not contain the Goldstone mode $\theta(x)$ anymore!

Translate to new fields with spontaneously broken symmetry

(595) $$\rho = \chi + v \quad \text{and} \quad \langle 0 | \chi | 0 \rangle = 0$$

Plugging in this into $\mathcal{L}$ yields:

(596) $$\mathcal{L} = \cdots + \boxed{\frac{1}{2} g^2 v^2 A_\mu A^\mu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

mass term for gauge field!

(597) $$\boxed{m_A^2 = g^2 v^2} \quad \text{mass}$$

Note: it was important that we had fixed the gauge so that there was no Goldstone mode left in the Lagrangian

Solves fundamental problem of Young-Mills theory
→ Nobel prize 2014

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