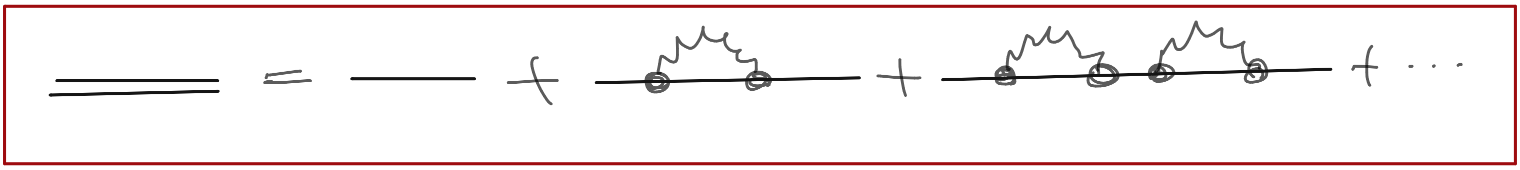
VI.5 Mass- and Charge Renormalization Ward identities

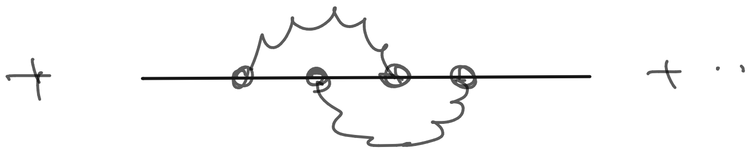
(A) Mass renormalization

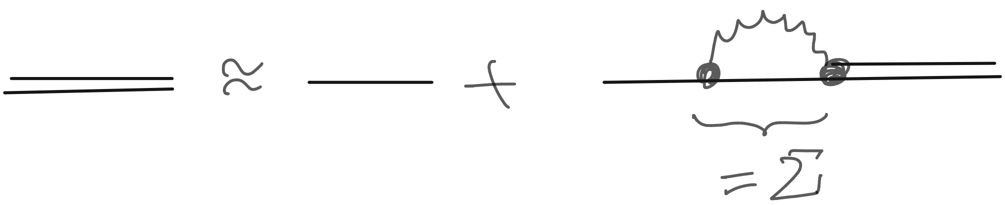
We have seen that the partial summation of all self energy diagrams has lead to the dressed electron propagator with pole $p^2 = \bar{m}^2$

$$m + \Sigma(\bar{m}) = \bar{m}$$

Only $\bar{m}$ is observable and thus a physical quantity. To simplify the notation we will use "m" again but for the renormalized mass.







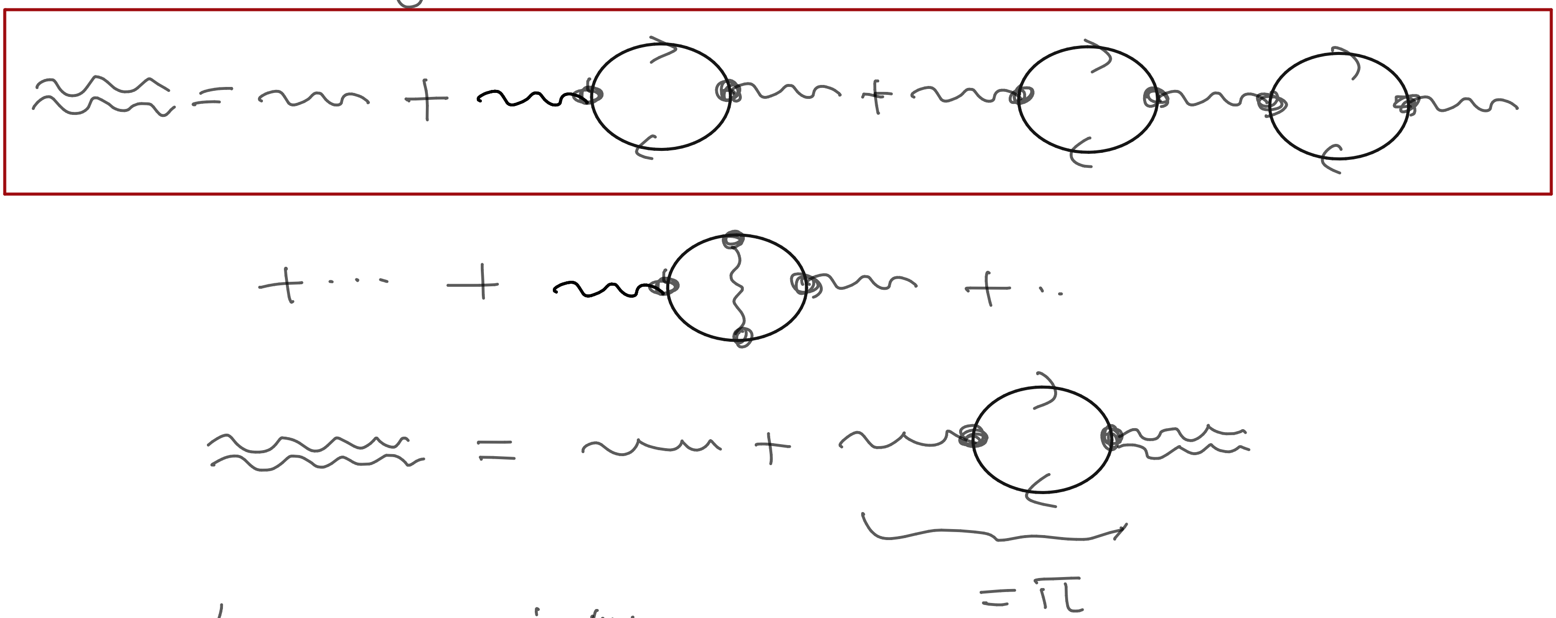
$$S(p) \to S'(p) = \frac{Z_2}{\not{p} - \underbrace{(m + \Sigma(\bar{m}))}_{=\bar{m}} - Z_2 \Sigma_R(\not{p}) + i\varepsilon} \qquad (537)$$

This also gave a contribution to the charge renormalization

$$e_0 \longrightarrow Z_2 e_0 \tag{532}$$

(B) Charge renormalization

We have seen that the partial resummation of all diagrams to the vacuum polarization has lead to the dressed photon propagator which did not produce a mass but another contribution to the charge renormalization



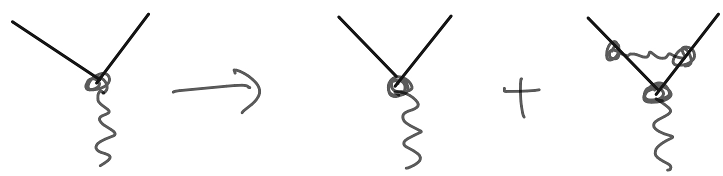
$$D'^{\mu\nu}(q) = \frac{i g^{\mu\nu}}{-q^2 + i\varepsilon} \frac{Z_3}{1 + Z_3 \Pi(q^2)} \tag{533}$$

where $\Pi(q^2 = 0) = 0$

This gave rise to

$$e_0 \to \sqrt{Z_3} Z_2 e_0$$

Finally we have seen that the vertex correction



$$\gamma^\mu \longrightarrow \gamma^\mu + \Lambda^\mu(p,q) = \frac{1}{Z_1}\gamma^\mu + \text{regular terms that vanish for } q^2 \to 0 \tag{534}$$

This gave rise to another contribution to the charge renormalization

$$\sqrt{Z_3}\, Z_2\, e_0 \longrightarrow \boxed{\sqrt{Z_3}\,\frac{Z_2}{Z_1} = e} \tag{535}$$

One can show that the factors Z_1, Z_2, Z_3 can be absorbed into the charge e in all orders of perturbation in the same unique way, eq. (535)

However: Does the charge renormalization depend on the type of fermion? Z_1 and Z_2 contain the fermion mass! So why is the proton charge $= |e|$?

(C) Ward identities

The last question can be answered using the

(536) Ward-Takahashi identity

$$\boxed{\Lambda^\mu(P,0) = -\frac{\partial}{\partial p_\mu}\Sigma_1(p)}$$

Note that for $q = p' - p = 0 \Rightarrow p' = p$ and thus

$$P = \frac{1}{2}(p+p') = p$$

proof:

$$\Lambda^\mu(P,q) = i e_0^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\nu \frac{1}{m - \not{p}' + \not{k} - i\varepsilon} \gamma^\mu \frac{1}{m - \not{p} + \not{k} - i\varepsilon} \gamma_\nu \cdot \frac{1}{\lambda^2 - k^2 - i\varepsilon}$$

and

$$\Sigma(p) = -i e_0^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\nu \frac{1}{m - \not{p} - \not{k} - i\varepsilon} \gamma_\nu \frac{1}{\lambda^2 - k^2 - i\varepsilon}$$

(later $\lambda = 0$). Now

$$\frac{1}{m - \not{p}' + \not{k}} - \frac{1}{m - \not{p} + \not{k}} = \frac{1}{m - \not{p}' + \not{k}} (\not{p}' - \not{p}) \frac{1}{m - \not{p} + \not{k}}$$

$$q_\mu \Lambda^\mu\left(\frac{p+p'}{2}, q\right) = \Sigma(p) - \Sigma(p')$$

i.e. in the limit $q \to 0$, i.e. $p' \to p$ one has

$$\Sigma(p') = \Sigma(p) + \frac{\partial \Sigma(p)}{\partial p_\mu} \underbrace{(p'-p)_\mu}_{= q_\mu} + \cdots$$

$$\Rightarrow \quad \Lambda^\mu(p,0) = -\frac{\partial \Sigma(p)}{\partial p_\mu} \qquad \square$$

<u>rem.:</u> One can show that this holds in all orders of magnitude

The Ward identity has an important consequence: In the low-energy limit $p^2 \to m^2$ we find

$$\Lambda^\mu(p,0) = \gamma^\mu \left(\frac{1}{Z_1} - 1\right)$$

on the other hand $(\delta m = \bar{m} - m)$

$$\Sigma(p) = \delta m + (m - \not{p})\left(\frac{1}{Z_2} - 1\right)$$

using $\Lambda^\mu(p,0) = -\frac{\partial}{\partial p_\mu} \Sigma(p) \Rightarrow$

(537) $$\boxed{Z_1 = Z_2}$$

Thus the charge renormalization is just

(538) $$\boxed{e_0 \longrightarrow e = \sqrt{Z_3}\, e_0}$$

The renormalization of charge in QED is universal and does not depend on the mass of the fermion.

VI.6 Renormalizability of quantum field theories

Divergencies emerge in QFT's due to integration $\int d^d k$ in loop diagrams

Simplification: theories without derivatives in the (nonlinear) interaction Lagrangian (e.g. QED) ↳

consider typical diagrams

ℓ : # of internal loops

n_B : # of internal boson lines

n_F : # of internal fermion lines

global degree of divergence $\quad$ d : dimension of space

D = power of k in numerator − power of k in denominator

$$(539) \quad \boxed{D = \ell d - 2n_B - n_F}$$

if $D > 0$ diagram divergent

$D = 0$ diagram log-divergent

$D < 0$ diagram convergent

more precisely: a given diagram is convergent if all its sub-diagrams are convergent

Question: Can we read off the degree of divergence of all diagrams and diagram parts just from the interaction Lagrangian?

consider: $\mathcal{L}_I \sim (\bar{\psi} \Theta \psi)^{F/2} \phi^B$

F: # of fermion fields per vertex (even)
B: # of boson fields per vertex

N_B: # of external boson lines
N_F: # of external fermion lines

n: order of perturbation

Since each propagator couples to two vertices:

$$N_B + 2n_B = nB \tag{540}$$

$$N_F + 2n_F = nF \tag{541}$$

number of 4-momenta not fixed by external lines = # of loops ℓ

$$\ell = n_B + n_F - n + 1 \tag{542}$$

Each internal line has a free momentum and each vertex gives one constraint except that the total momentum is not fixed by the vertices

$$\boxed{D = n\underline{I} + \underbrace{d - \frac{1}{2}(d-2)N_B - \frac{1}{2}(d-1)N_F}_{\text{depends on specific diagram}}} \tag{543}$$

order of perturbation

divergency index

$$\boxed{\underline{I} = \frac{1}{2}(d-2)B + \frac{1}{2}(d-1)F - d} \tag{544}$$

depends only on dimension of space d and interaction Lagrangian

$\boxed{d=4}$

$$\underline{I} = B + \frac{3}{2}F - 4 \tag{545}$$

$$D = nI + 4 - N_B - \frac{3}{2} N_F \tag{546}$$

$\Longrightarrow$

- $I > 0$ Starting from a certain order of perturbation all diagrams are divergent

  non-renormalizable theory

- $I < 0$ Only a finite number of sub-diagrams in a finite number of diagrams are divergent

  super-renormalizable theory

- $I = 0$ D is independent on n (order of perturbation). In $d=4$ $D<0$ if $N_B + \frac{3}{2} N_F > 4$

  $\Rightarrow$ Only a small number of subdiagrams lead to divergent diagrams in all order perturbation theory

  Superficially renormalizable theory

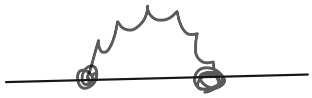
example QED $\quad B = 1 \quad F = 2$

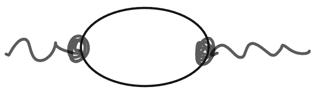
$$I = 1 + \frac{3}{2} 2 - 4 = 0 \tag{547}$$

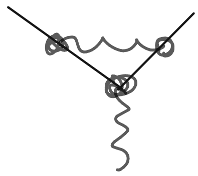
$\Rightarrow$ QED is superficially renormalizable

$\exists$ only finite number of sub-diagrams for which

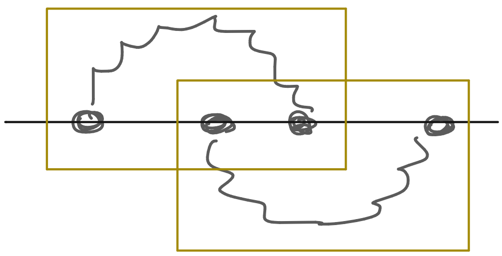
$$4 - N_B - \frac{3}{2} N_F \geq 0$$

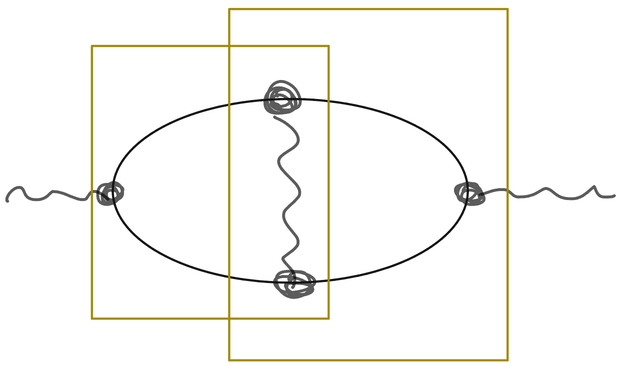
self energy  $D = 4 - 0 - \frac{3}{2} 2 = 1$

vacuum polarization  $D = 4 - 2 = 2$

Vertex correction  $D = 4 - 1 - \frac{3}{2} 2 = 0$

To prove renormalizability of a superficially renormalizable theory one has to take into account that there are overlapping divergent diagrams such as





Using Ward identities one can show that all overlapping diagrams can be reduced to non-overlapping ones!

$\Rightarrow$ **proof of renormalizability of QED**