VI.3 Dimensional regularization

With the selfenergy of the electron and the vacuum polarization we have seen two important examples of loop diagrams and have seen the principle ideas behind the elimination of divergencies. Now we want to discuss a general procedure for the systematic separation of infinite contributions from finite contributions of a loop diagram.

The general structure of loop-diagram expressions is:

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{N}{A_1 A_2 \cdots A_n} \tag{487}$$

where

A_j : denominator of propagators
$k^2 + i\varepsilon \qquad k^2 - m^2 + i\varepsilon$

N : polynom in k

There are two steps in the regularization

(i) write integrals I in normal form

The main problem in (487) is the denominator:

$$\frac{1}{A_1 A_2} = \int_0^1 dz \frac{1}{[A_1 z + A_2(1-z)]^2} = \int_0^1 dz \frac{1}{[D(z)]^2} \tag{488}$$

$$\text{with} \quad D(z) \equiv A_1 z + A_2(1-z) \tag{489}$$

[proof: $f(z) = \frac{1}{A_1 z + A_2(1-z)}$ $\quad \frac{df}{dz} = \frac{-A_1 + A_2}{(A_1 z + A_2(1-z))^2}$

$$= (A_2 - A_1) f^2$$

$$\int_0^1 dz\, f(z)^2 = \frac{1}{A_2 - A_1} \int_{1/A_2}^{1/A_1} df = \frac{1}{A_2 - A_1}\left[\frac{1}{A_1} - \frac{1}{A_2}\right]$$

$$= \frac{1}{A_2 - A_1} \frac{A_2 - A_1}{A_1 A_2} = \frac{1}{A_1 A_2}$$]

similarly

$$\frac{1}{A_1 A_2 A_3} = 2\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \frac{1}{[A_1 z_1 + A_2 z_2 + A_3(1-z_2-z_3)]^3} \tag{490}$$

$$= 2\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \frac{1}{[D(z_1,z_2)]^3}$$

$$D(z_1,z_2) = A_1 z_1 + A_2 z_2 + A_3(1-z_2-z_3) \tag{491}$$

Since D is linear in the denominators of propagators is has the form

(492) $$D = B^2 + 2k\cdot Q - k^2$$

with variable transformation $k = k' - Q$

(493) $$D = B^2 + Q^2 - k'^2 = C^2 - k'^2$$

This shows that all loop integrals can be written in the form (apart from integrations over Feynman parameters $z_1, z_2, \dots$)

(494) $$\boxed{\int \frac{d^4k}{(2\pi)^4} \frac{N}{(C^2 - k^2 - i\varepsilon)^n}}$$

standard form of loop integrals

(ii) dimensional regularization

The integral (494) for $N = \text{const}$ is convergent for $n > 2$ but divergent for $n \leq 2$.

This is different in other dimensions

(494) for $N = \text{const}$ is convergent if $d < 2n$

$\Rightarrow$ consider general d and then take $d \to 4$. This will allow to separate divergent from convergent terms!

for $d < 2n$

$$\boxed{\int \frac{d^dk}{(2\pi)^d} \frac{1}{(c^2-k^2-i\varepsilon)^n} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{c^2}\right)^{n-\frac{d}{2}}}$$

(495)

$$\boxed{\int \frac{d^dk}{(2\pi)^d} \frac{k_\mu k^\nu}{(c^2-k^2-i\varepsilon)^n} = \frac{-i g^{\mu\nu}}{2(4\pi)^{d/2}} \frac{\Gamma(n-1-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{c^2}\right)^{n-1-\frac{d}{2}}}$$

(496)

Here

$$\boxed{\begin{aligned} &\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1) \\ &\Gamma(1) = 1 \end{aligned}}$$

Gamma function

for $\alpha = n \in \mathbb{N}$ $\quad \Gamma(n) = (n-1)!$

$$|\Gamma(0)| = |\Gamma(-1)| = |\Gamma(-2)| = \cdots = \infty$$

Γ is also well defined for non-integer arguments

(497)

$$\boxed{\Gamma(\alpha) = \int_0 dt\, t^{\alpha-1} e^{-t}}$$

Consider integrals I as function of the continuous parameter d and let $d \to 4$ at the end

Regularization of vacuum polarization

$$\Pi^{\mu\nu}(q^2) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\{\gamma^\mu(m+\not{p})\gamma_\nu(m+\not{q}-\not{p})\}}{(m^2-p^2-i\varepsilon)(m^2-(p-q)^2-i\varepsilon)} \tag{498}$$

Applying the rules for traces of γ matrices yields

$$\Pi^{\mu\nu}(q^2) = -4ie^2 \int \frac{d^4p}{(2\pi)^4} * \frac{\overbrace{m^2 g^{\mu\nu} + 2p^\mu p^\nu - p^\mu q^\nu - q^\mu p^\nu - g^{\mu\nu} p(p-q)}^{= N^{\mu\nu}}}{(m^2-p^2-i\varepsilon)(m^2-(p-q)^2-i\varepsilon)}$$

this can be brought into normal form

$$\frac{1}{(m^2-p^2-i\varepsilon)(m^2-(p-q)^2-i\varepsilon)} = \int_0^1 dz \frac{1}{[(m^2-p^2)(1-z)+(m^2-(p-q)^2)z]^2}$$

$\Rightarrow$ in d-dimensions

$$\Pi_d^{\mu\nu}(q^2) = -4ie^2 \int_0^1 dz \int \frac{d^dp}{(2\pi)^d} \frac{N^{\mu\nu}}{[m^2-p^2+2pqz-q^2z-i\varepsilon]^2}$$

substitution $\quad p = k + qz$

$$\Pi_d^{\mu\nu}(q^2) = -4ie^2 \int_0^1 dz \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{N}^{\mu\nu}}{[m^2-k^2-q^2z(1-z)-i\varepsilon]^2}$$

where
$$\tilde{N}^{\mu\nu} = m^2 g^{\mu\nu} + 2k^\mu k^\nu + (q^\mu k^\nu + k^\mu q^\nu)(2z-1)$$
$$-2q^\mu q^\nu (1-z)z - g^{\mu\nu}(k+qz)(k-q(1-z))$$

<u>note:</u>
- terms with odd powers of k^μ vanish because the symmetric integration area
- for the same reason only diagonal terms $k^\mu k^\nu \longrightarrow g^{\mu\nu}\frac{k^2}{d}$ survive

so

$$\tilde{N}^{\mu\nu} \longrightarrow g^{\mu\nu}\left(m^2 + \left(\frac{2}{d}-1\right)k^2 + q^2 z(1-z)\right) - 2q^\mu q^\nu z(1-z)$$

Now we can carry out the integration $\int d^dk$. This yields

$$\Pi_d^{\mu\nu}(q^2) = \frac{4e^2}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \int_0^1 dz \frac{1}{(m^2 - q^2 z(1-z))^{2-d/2}}$$
$$\cdot \Big[g^{\mu\nu}(m^2 - q^2 z(1-z)) \underbrace{\left(1 - \frac{d}{2}\left(\frac{2}{d}-1\right)\frac{\Gamma(1-d/2)}{\Gamma(2-d/2)}\right)}_{=0}$$
$$+ (g^{\mu\nu}q^2 - q^\mu q^\nu) 2z(1-z) \Big]$$

$\Rightarrow$

(499) $$\Pi_d^{\mu\nu}(q^2) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi_d(q^2)$$

This also proofs eq. (479). Here ($e^2=\alpha$)

(500) $$\Pi_d(q^2) = \frac{\alpha}{\pi} \frac{\Gamma(3-\frac{d}{2})}{(2-\frac{d}{2})(4\pi)^{d/2-2}} \int_0^1 dz \frac{2z(1-z)}{\left(m^2-q^2 z(1-z)\right)^{2-d/2}}$$

This expression is well behaved for all $d<4$.

Let us now consider the limit $d \to 4$

(501) $$\underbrace{\Pi_d(0)}_{\text{divergent for } d\to 4} + \underbrace{\bar{\Pi}_d(q^2)}_{\text{finite for } d\to 4} = \Pi_d(q^2)$$

(502) $$\boxed{\Pi_d(0) = \frac{\alpha}{\pi} \frac{2\Gamma(1+\varepsilon/2)}{\varepsilon(4\pi)^{\varepsilon/2}} \int_0^1 dz \frac{2z(1-z)}{m^\varepsilon} \xrightarrow[\varepsilon\to 0]{} \infty \qquad \varepsilon \equiv 4-d}$$

This is the term which will contribute to Z_3 and can be ignored.

$$\bar{\Pi}_d(q^2) = \frac{\alpha}{\pi} \frac{4\Gamma(1+\frac{\varepsilon}{2})}{\varepsilon(4\pi)^{-\varepsilon/2}} \int_0^1 dz\, z(1-z) \cdot \left[\frac{1}{[m^2-q^2 z(1-z)]^{\varepsilon/2}} - \frac{1}{m^\varepsilon}\right]$$

Using $\lim\limits_{\varepsilon\to 0} A^{\varepsilon} = 1 + \varepsilon \ln A + \mathcal{O}(\varepsilon^2)$

$\Rightarrow$ $(\varepsilon \to 0)$

$$\boxed{\bar{\Pi}(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dz\, z(1-z)\ln\left[1-\frac{q^2}{m^2}z(1-z)\right]}$$

(503) finite contribution to the vacuum polarization

in the low energy regime, i.e. for $|\vec{q}| \ll m$

(504) $$\boxed{\bar{\Pi}(q^2) \longrightarrow \frac{\alpha}{15\pi}\frac{q^2}{m^2}}$$

The modification of the photon propagator

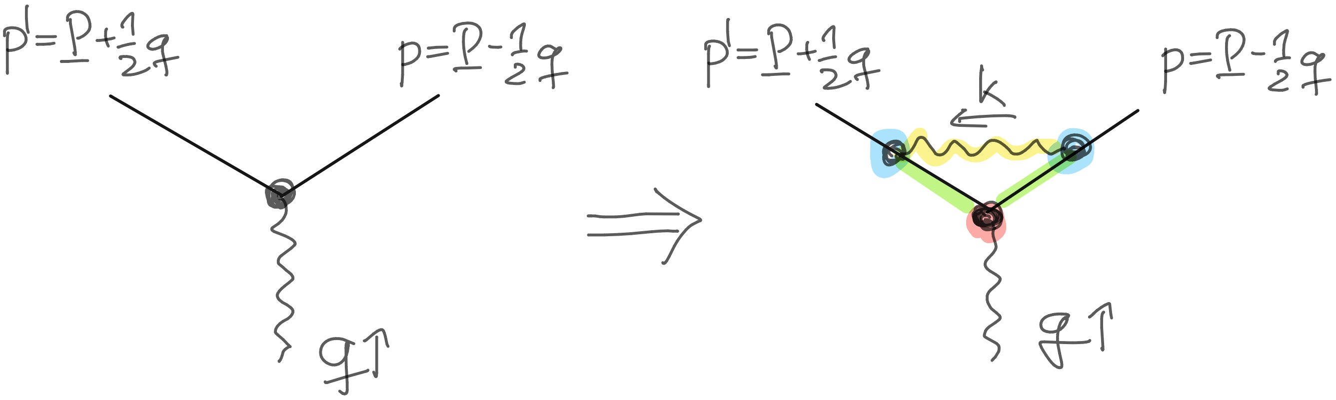
(505) $$D^{\mu\nu}(q^2) \longrightarrow D'^{\mu\nu}(q^2) = \frac{g^{\mu\nu}}{-q^2-i\varepsilon}\frac{1}{1+\bar{\Pi}(q^2)}$$

leads to a modification of the Lamb shift next to the Bethe result

$\Rightarrow \ldots \Rightarrow$ for Hydrogen

$$\Delta E_{1S,2S} \simeq -27\ \mathrm{MHz}$$

VI.4 Vertex correction and anomalous magnetic moment



$$-e\bar{u}(p')\gamma^\mu u(p) \Longrightarrow -e\bar{u}(p')\Lambda^\mu(p,q)u(p)$$

(506) $P=\frac{1}{2}(p+p')$ $\qquad$ $q=p'-p$

average 4-momentum $\qquad$ relative 4-momentum

(507)
$$\Lambda^\mu(p,q)=ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\overbrace{\gamma^\nu(m-\not{p}'-\not{k})\gamma^\mu(m-\not{p}-\not{k})\gamma_\nu}^{=N^\mu}}{(m^2-(p'-k)^2-i\varepsilon)(m^2-(p-k)^2-i\varepsilon)(\lambda^2-k^2-i\varepsilon)}$$

vertex correction

Here we have introduced a ficticious photon mass λ which we will set $\lambda=0$ later

transformation to normal form

(508) $$A_1 = m^2 - (p'-k)^2 - i\varepsilon = 2p'k - k^2 - i\varepsilon$$

We assume here that the outgoing fermion is a free fermion for which $p'^2 - m^2 = 0$. The same is assumed for the incoming fermion i.e. $p^2 - m^2 = 0$

(509) $$A_2 = m^2 - (p-k)^2 - i\varepsilon = 2pk - k^2 - i\varepsilon$$

(510) $$A_3 = \lambda^2 - k^2 - i\varepsilon$$

- denominator of Λ^μ in (507)

$$\frac{1}{A_1 A_2 A_3} = 2\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \frac{1}{\underbrace{[A_1 z_1 + A_2 z_2 + A_3(1-z_1-z_2)]}_{=D(z_1,z_2)}{}^3}$$

i.e.

(511) $$D(z_1,z_2) = 2(z_1 p' + z_2 p)k + \lambda^2(1-z_1-z_2) - k^2 - i\varepsilon$$

variable trafo: $k = k' + z_1 p' + z_2 p$ removes linear term in k', i.e.

$$D(z_1,z_2) = (z_1+z_2)^2 m^2 - z_1 z_1 q^2 + \lambda^2(1-z_1-z_2) - k'^2 - i\varepsilon$$

where we have used again that $m^2 = p^2 = p'^2$ and $q = p' - p$. (remember $p = (p_0, \vec{p})$ $p' = (p_0', \vec{p}')$ and $p^2 = p_0^2 - |\vec{p}|^2$)

- Numerator after substitution $k \to k'$

$$N^\mu = \gamma^\nu (m + \not{p}'(1-z_1) - z_2 \not{p} - \not{k}') \gamma^\mu (m + \not{p}(1-z_2) - \not{p}' z_1 - \not{k}') \gamma_\nu$$

$$= \gamma^\nu (m + \not{p}'(1-z_1) - z_2\not{p}) \gamma^\mu (m + \not{p}(1-z_2) - \not{p}' z_1) \gamma_\nu$$

(5.12) $$+ \gamma^\nu \not{k}' \gamma^\mu \not{k}' \gamma_\nu + \underbrace{\text{linear terms in } \not{k}'}_{\hookrightarrow\, 0 \text{ after integration } \int d^4k'}$$

Some further simplifications:

- $\gamma^\lambda \not{a} \gamma_\lambda = -2\not{a} \qquad \gamma^\lambda \not{a}\not{b}\gamma_\lambda = 4\, a \cdot b$

  $\gamma^\lambda \not{a}\not{b}\not{c}\gamma_\lambda = -2\not{c}\not{b}\not{a}$

- terms $\sim (z_2 - z_1)^1$ vanish in integrals $\int dz_1 \int dz_2$
- terms $\sim (\not{q})^1$ vanish after sandwiching Λ^μ with $\bar{u}(p')$ and $u(p)$
- in integral $\int d^4k'$: $k'^\mu k'^\nu \to g^{\mu\nu} k^2/4$

This then leads to $\cdots \Rightarrow$

(5.13) $$\boxed{\Lambda^\mu (P, q) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)}$$

(5.14) where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ (see Dirac theory)

$F_1(q^2)$ and $F_2(q^2)$ are scalar functions of q^2

$$\boxed{\begin{aligned} &(5.15)\quad F_1(q^2) = 2ie^2 \int \frac{d^4k'}{(2\pi)^4} \int_0^1 dz_1 \int_0^{1-z_1} dz_2 \, \frac{N_1}{(D[k'])^3} \\ &(5.16)\quad F_2(q^2) = -2ie^2 \int \frac{d^4k'}{(2\pi)^4} \int_0^1 dz_1 \int_0^{1-z_1} dz_2 \, \frac{4m^2(z_1+z_2)(1-z_1-z_2)}{(D[k'])^3} \end{aligned}}$$

(5.17) $N_1 = k'^2 - 2q^2(1-z_1)(1-z_2) - 2m^2\left[1-4(1-z_1-z_2)+(1-z_1-z_2)^2\right]$

we see that the vertex correction

$$\gamma^\mu \longrightarrow \Gamma^\mu(P, q)$$

$$P = \frac{1}{2}(p+p') \qquad q = q' - q$$

has two effects

(i) modification of ordinary vertex factor $\sim \gamma^\mu$ which accounts for the coupling of the elm. field to the Dirac current

(ii) new current term $\sim i\sigma^{\mu\nu}q_\nu$
$\Rightarrow$ anomalous or Pauli current

One recognizes

$$F_2 \sim \int d^4k' \, \frac{1}{k'^6} < \infty \qquad \text{regular}$$

$$F_1 \sim \int d^4k' \, \frac{k'^2}{k'^4} \longrightarrow \infty \qquad \text{divergent}$$

renormalization of F_1: full charge renormalization

One recognizes that $F_1(q^2) - F_1(0)$ is regular i.e.

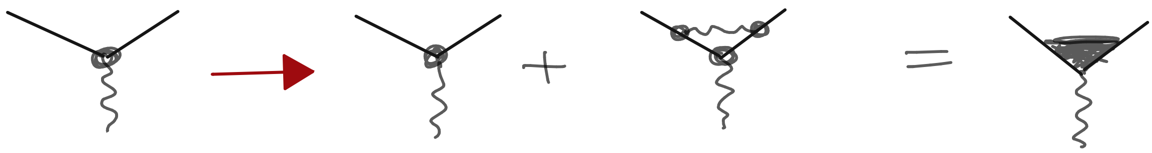
(5.18)
$$\Lambda^\mu(P,q) = \underbrace{\Big(F_1(q^2) - F_1(0)\Big)\gamma^\mu + F_2(q^2)\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}}_{\text{regular part of dressed vertex}}$$

$$+ \underbrace{F_1(0)\gamma^\mu}_{} \Leftarrow \text{divergent part}$$

new renormalization constant Z_1

(5.19)
$$\boxed{F_1(0) = \frac{1}{Z_1} - 1}$$

vertex:



$$\gamma^\mu \longrightarrow \underbrace{\gamma^\mu + \Lambda^\mu(P,q)}_{= \Gamma^\mu(P,q)} \qquad \Gamma^\mu$$

$$\Gamma^\mu(p,q) = \underbrace{\left[F_1(q^2) - F_1(0)\right]\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}}_{\text{regular}} + \underbrace{\frac{1}{Z_1}\gamma^\mu}_{\text{divergent}} \quad (520)$$

and $\to 0$ for $q^2 \to 0$

full charge renormalization in $\gamma^\mu \to \Gamma^\mu$

$$\boxed{e_0 \longrightarrow \underset{\text{from wavefunction renormalization}}{e_0\sqrt{Z_3}Z_2} \longrightarrow e_0\sqrt{Z_3}\frac{Z_2}{Z_1}} \quad (521)$$

The anomalous magnetic moment of the electron

One important result of QED is the precise prediction of the magnetic moment of the electron observed when an electron is subject to an external elm. (magnetic) field

Dirac equation in external field A_μ

$$(i\gamma^\mu\partial_\mu - eA_\mu\gamma^\mu - m)\,\psi = 0 \quad (522)$$

Now for external elm. field $q^2 = q^{0\,2} - |\vec{q}\,|^2 \overset{!}{=} 0$

With regular part of vertex correction

$$e\gamma^\mu A_\mu \rightarrow e\Gamma^\mu A_\mu = e\gamma^\mu A_\mu + eF_2(0)\frac{i\sigma^{\mu\nu}q_\nu}{2m}A_\mu$$

(note $q^2=0$ does not mean $q^\nu=0$). This expression is in Fourier space. In real space

$$iq_\nu A_\mu \rightarrow \frac{\partial}{\partial x^\nu}A_\mu$$

so

$$e\gamma^\mu A_\mu \rightarrow e\gamma^\mu A_\mu + eF_2(0)\frac{\sigma^{\mu\nu}A_{\mu,\nu}}{2m}$$

$$= e\gamma^\mu A_\mu + eF_2(0)\frac{\sigma^{\mu\nu}F_{\mu\nu}}{4m}$$

since $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$

where $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ is the field strength tensor. This gives for the Dirac equation with vertex correction

(523) $$\left(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - eF_2(0)\frac{\sigma^{\mu\nu}F_{\mu\nu}}{4m} - m\right)\psi = 0$$

This correction term modifies the magnetic moment of the electron

(524) $$\boxed{\vec{\mu}_{mag} = g\frac{e}{2m}\vec{S}}$$

Dirac theory $\boxed{g=2}$

QED $\boxed{g=2+\kappa}$

$$\kappa = F_2(0) + \text{higher order terms}$$

$$F_2(0) = -2ie^2 \int_0^1 dz_1 \int_0^{1-z_1} dz_2 \,*$$

(525) $$* \int \frac{d^4k'}{(2\pi)^4} \frac{4m^2(z_1+z_2)(1-z_1-z_2)}{[\underbrace{m^2(z_1+z_2)^2+\lambda^2(1-z_1-z_2)}_{\equiv B^2}-k'^2-i\varepsilon]^3}$$

With the integrals (495) and (496) we find

(526) $$\int \frac{d^4k'}{(2\pi)^4} \frac{1}{[B^2-k'^2-i\varepsilon]^3} = \frac{i}{32\pi^2 B^2}$$

With this and setting the ficticious photon mass $\lambda=0$

(527) $$F_2(0) = \frac{2e^2}{32\pi^2} 4m^2 \int_0^1 dz_1 \int_0^{1-z_1} dz_2 \frac{\cancel{(z_1+z_2)}(1-z_1-z_2)}{m^2(z_1+z_2)^{\cancel{2}}+\underbrace{\cancel{\lambda^2(1-z_1-z_2)}}_{=0}}$$

Set
$$\xi = z_1+z_2 \qquad \eta = \frac{z_1-z_2}{2(z_1+z_2)} = \frac{z_1-z_2}{2\xi}$$

and use:
$$\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \longrightarrow \int_0^1 d\xi \int_{-1/2}^{1/2} d\eta\, \xi$$

$$F_2(0) = \frac{e^2}{4\pi}\int_0^1 d\xi \int_{-1/2}^{1/2} dy\,(1-\xi) = \frac{e^2}{\pi}\int_0^1 d\xi\,(1-\xi)$$

(528)
$$= \frac{e^2}{2\pi} = \underline{\underline{\frac{\alpha}{2\pi}}}$$ J. Schwinger 1948

(529)
$$\boxed{g-2 = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2) = 0.0011614}$$

QED result of anomalous magnetic moment

<u>theory up to α^4:</u>

$$g-2 = \frac{\alpha}{2\pi} - 0.328478966\left(\frac{\alpha}{\pi}\right)^2 + 1.17611(42)\left(\frac{\alpha}{\pi}\right)^3 - 1.434(136)\left(\frac{\alpha}{\pi}\right)^4$$

$$\boxed{\begin{aligned}(g-2)_{\text{theory}} &= 0.00115965214\,0(28)\\ (g-2)_{\text{exp.}} &= 0.00115965218\,84(43)\end{aligned}}$$

$$\frac{\Delta x}{x} \sim 10^{-12}$$ phantastic agreement !