VI. Renormalization of QED

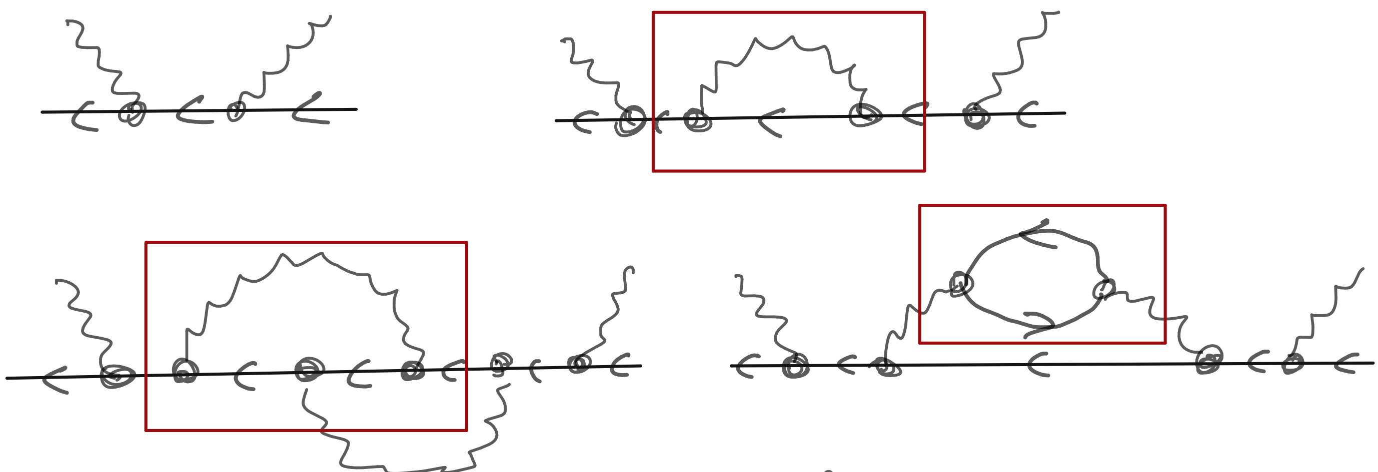
We will now consider the question of how to treat the loop diagrams that emerge in higher order perturbation theory. They are formally divergent but we will see that they nevertheless lead to resonable and finite results as we can systematically remove these divergencies. This will be done in two steps

(i) Regularization = separation of divergencies

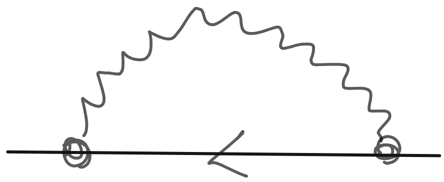
(ii) Renormalization = formal absorption of divergent expressions in the parameters of the theory such as m, e

example

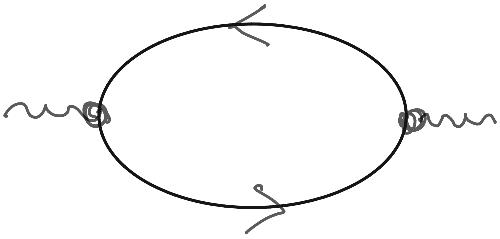
Compton scattering, typical diagrams



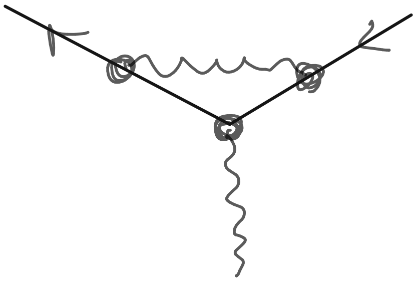
We here see 3 types of loop diagrams:

(457) 

self energy of the fermion (electron)

(458) 

vacuum polarization

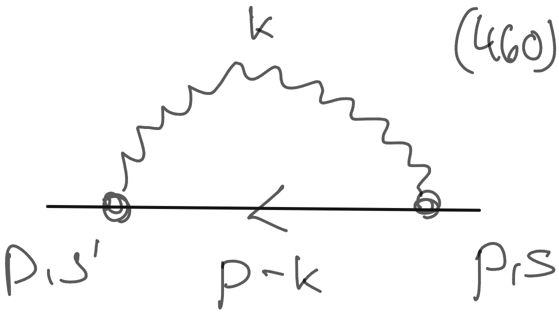
(459) 

vertex correction

For QED these are the only types of loop diagrams and we will discuss them one by one

VI.1 Selfenergy of the electron

(460)



$$\mathcal{M}_{\Sigma} = \bar{u}(p,s')\,\Sigma(p)\,u(p,s)$$

according to Feynman rules

$$\boxed{\Sigma_2(p) = -ie^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\mu(m+\not p-\not k)\gamma_\mu}{(-k^2-i\varepsilon)(m^2-(p-k)^2-i\varepsilon)}} \tag{461}$$

<u>remember:</u> integration is over d^4k, i.e. also for values of $k^0 \neq |\vec{k}|$, i.e.

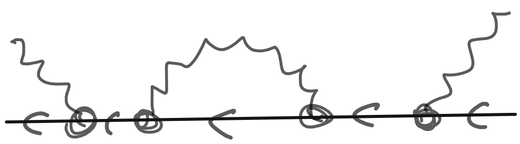
"off energy shell"

One recognizes that this integral diverges as $|k| \to \infty$

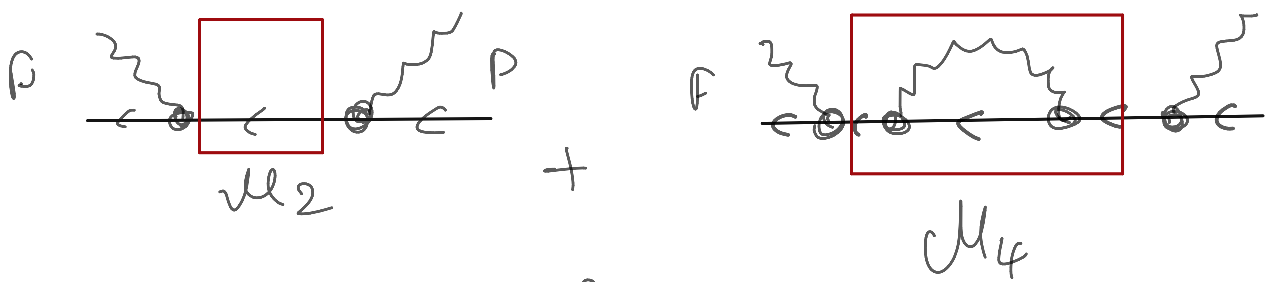
$$\int d^4k \,\frac{\not k}{k^4} \sim \int^{\infty} d|k| \to \infty$$

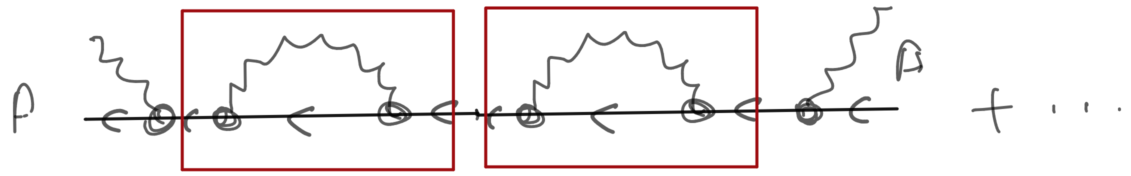
Before discussing the renormalization of $\Sigma(p)$ we want to analyze its physical meaning:

For a diagram of the type



there exists higher order terms of the same structure





Summing up of all these terms (which is of course only a partial summation over all possible diagrams) yields

$$\mathcal{M}_2 = -ie^2\, \bar{u}(p')\, \not{\epsilon}_f\, iS(p)\, \not{\epsilon}_i\, u(p)$$

$$\mathcal{M}_4 = -ie^2\, \bar{u}(p')\, \not{\epsilon}_f\, iS(p)\big(-i\Sigma(p)\big)\, iS(p)\, \not{\epsilon}_i\, u(p)$$

$$\mathcal{M}_6 = -ie^2\, \bar{u}(p)\, \not{\epsilon}_f\, iS(p)\big(-i\Sigma(p)\big)\, iS(p)\big(-i\Sigma(p)\big) \cdot iS(p)\, \not{\epsilon}_i\, u(p)$$

$\vdots$ etc

(462) $$\mathcal{M} = \sum_{n=1}^{\infty} \mathcal{M}_{2n} = -ie^2\, \bar{u}(p)\, \not{\epsilon}_f\, iS'(p)\, \not{\epsilon}_i\, u(p)$$

where

(463) $$S'(p) = S(p)\left[1 + \Sigma(p)S(p) + \Sigma(p)S(p)\Sigma(p)S(p) + \dots\right]$$

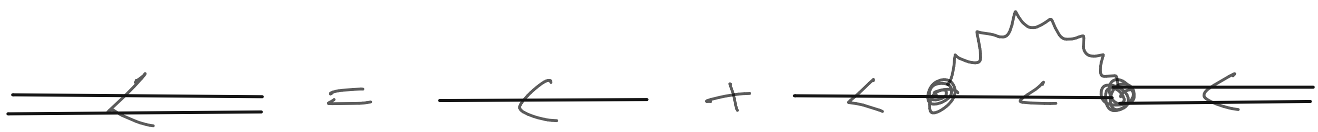
This leads to the Dyson equation of the e-propagator

(464) $$\boxed{S'(p) = S(p) + S(p)\,\Sigma_i(p)\,S'(p)}$$

where

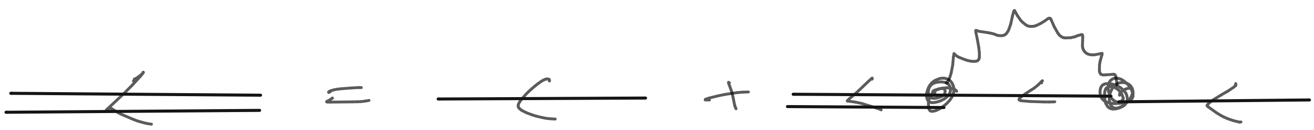
$$S(p) = \frac{1}{\not{p} - m + i\varepsilon} = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}$$

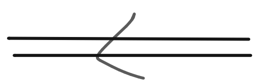
graphic representation



or alternatively with a different ordering of terms

$$\boxed{S'(p) = S(p) + S'(p)\Sigma(p)S(p)} \qquad (465)$$



 is calle the dressed electron propagator

remark: We will see later that there are other contributions that lead to a dressing of the propagator

formal solution of (464) or (465) is

$$S'(p) = S(p)\frac{1}{1 - \Sigma(p)S(p)} = \frac{1}{1 - S(p)\Sigma(p)}S(p) \qquad (466)$$

$\Sigma(p)$ is a 4×4 Dirac matrix and a Lorentz scalar, i.e. it can be written as

(467) $$\Sigma(p) = \underbrace{m\,A(p^2)}_{\text{unit matrix}} + \underbrace{\not p\,B(p^2)}_{\text{all } \gamma \text{ matrices}}$$

where $A(p^2)$ and $B(p^2)$ are scalars. This is the only possible decomposition as the only possibility to get a Lorentz scalar out of the γ-matrices and the 4-momentum is $\not p$.

Furthermore

(468) $$\left[\Sigma(p), S(p)\right] = 0$$

Now $$S'(p) = \frac{1}{S^{-1}(p) - \Sigma(p)} = \frac{1}{\not p - m - \Sigma(p) + i\varepsilon}$$

We will show that $\Sigma(p)$ has two effects:

(i) modification of the mass $m \to \bar{m}$

(ii) modification of normalization of the propagator

Lets expand $\Sigma(p)$ in a Taylor series of $(\not p - \bar{m})$ where $\bar{m}$ still needs to be determined

$$\Sigma(p) = \Sigma(\not p) = \Sigma(\bar{m}) + (\not p - \bar{m})\Sigma'(\bar{m}) + \frac{1}{2}(\not p - \bar{m})^2\Sigma''(\bar{m}) + \dots$$

$$= \underbrace{\Sigma(\bar{m})}_{4\times 4 \text{ matrix}} + \underbrace{(\not p - \bar{m})}_{4\times 4 \text{ matrix}}\Sigma'(\bar{m}) + \Sigma_R(\not p)$$

Note that $\Sigma'(\bar{m}) = \left.\frac{d\Sigma(p)}{d\not p}\right|_{\not p = \bar{m}}$ is already scalar

and the above notation is a very compact one:

$$\Sigma(\not p) = \begin{pmatrix} \Sigma_{11}(\not p) & \cdots & \Sigma_{14}(\not p) \\ \vdots & & \vdots \\ \Sigma_{41}(\not p) & \cdots & \Sigma_{44}(\not p) \end{pmatrix} = \Sigma(\bar{m}) +$$

$$+ \begin{pmatrix} (\not p - \bar{m})_{11}\frac{\partial \Sigma_{11}}{\partial \not p_{11}} + (\not p - \bar{m})_{12}\frac{\partial \Sigma_{11}}{\partial \not p_{12}} + \dots & \cdots \\ \vdots & \\ (\not p - \bar{m})_{11}\frac{\partial \Sigma_{41}}{\partial \not p_{11}} + (\not p - \bar{m})_{12}\frac{\partial \Sigma_{41}}{\partial \not p_{12}} + \dots & \cdots \end{pmatrix}$$

$\Rightarrow$

$$S'(p) = \frac{1}{\not p \underbrace{- m - \Sigma(\bar{m})}_{= -\bar{m}} - (\not p - \bar{m})\Sigma'(\bar{m}) - \Sigma_R(\not p) + i\varepsilon} \tag{469}$$

so

(470) $$\boxed{\bar{m} = m + \Sigma(\bar{m})}$$

Σ adds to the mass!

This gives

(471) $$S'(p) = \frac{1}{\underbrace{\not{p}\left(1-\Sigma'(\bar{m})\right)}_{=\frac{1}{Z_2}} - \bar{m}\underbrace{\left(1-\Sigma'(\bar{m})\right)}_{=1/Z_2} - \Sigma_R(\not{p}) + i\varepsilon}$$

(472) $$\boxed{Z_2 \equiv \frac{1}{1-\Sigma'(\bar{m})}}$$

divergent

which gives eventually

(473) $$\boxed{S'(p) = \frac{Z_2}{\not{p} - \bar{m} - Z_2 \Sigma_R(\not{p}) + i\varepsilon}}$$

lookin at the free electron propagator

$$S(p) = \frac{1}{\not{p} - m + i\varepsilon} = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}$$

we recognize that m is the pole of $S(p)$. We now show that the pole of $S'(p)$ in eq. (473) is $\bar{m}$

and thus $\bar{m}$ has to be interpreted as

renormalized mass

Note that $\Sigma_R(\not{p}) = (\not{p}-\bar{m})^2 R$ and $R \neq 0$ for $\not{p}=\bar{m}$

$$S'(p) = \frac{(\not{p}+\bar{m})Z_2}{(p^2-\bar{m}^2)(1-(p-\bar{m})Z_2 R)}$$

so obviously $S'(p)$ has a pole at $p^2=\bar{m}^2$!
Furthermore in the vicinity of the pole, so for low momenta $\not{p} \cong \bar{m} + \mathcal{O}(p^2-\bar{m}^2)$

$$\Rightarrow \quad \boxed{S'(p) \xrightarrow[p^2 \approx \bar{m}^2]{} \frac{Z_2}{\not{p}-\bar{m}}} \tag{474}$$

Further procedure:

- identify $\bar{m} \equiv$ physical mass (see later)
- Z_2 eliminate from theory

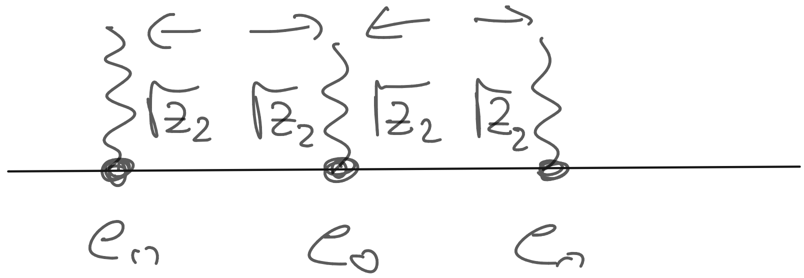
Elimination of Z_2: charge- and wavefunction renormalization

- all internal electron lines connect to two vertices

(475) $$Z_2 \longrightarrow \sqrt{Z_2}\sqrt{Z_2}$$

each vertex contains the electron charge e !

$\Rightarrow$ We can absorb the Z_2 factors of all internal lines in the charge e



$$e_0 \to \sqrt{Z_2}\, e_0 \sqrt{Z_2} = Z_2 e_0 \overset{!}{=} e$$

remark: this gives one contribution to the charge renormalization another one will follow later

- in external electron / positron lines a factor of $\sqrt{Z_2}$ is missing however

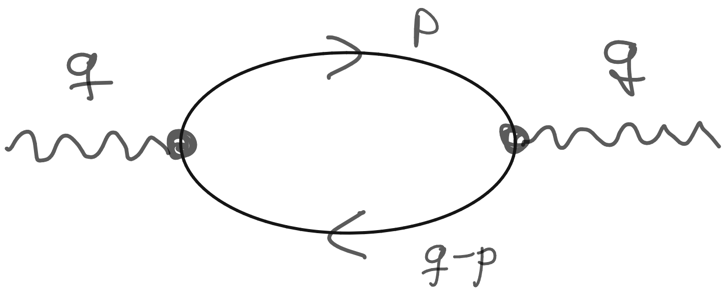
rule (10) for Feynman diagrams

$\boxed{\text{multiply every external fermion line with } \sqrt{Z_2}}$

wavefunction renormalization

physical background: incoming fermions can be considered to be created in some QED process in the past!

VI.2 Vacuum polarization



$$\mathcal{M}_\Pi = (\epsilon_f)^*_\mu \, \Pi^{\mu\nu}(q) \, (\epsilon_i)_\nu \tag{476}$$

according to Feynman rules

$$\boxed{\Pi^{\mu\nu}(q) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\{\gamma^\mu(m+\not{p})\gamma^\nu(m+\not{q}-\not{p})\}}{(m^2-p^2-i\varepsilon)(m^2-(p-q)^2-i\varepsilon)}} \tag{477}$$

-1 since closed fermion loop

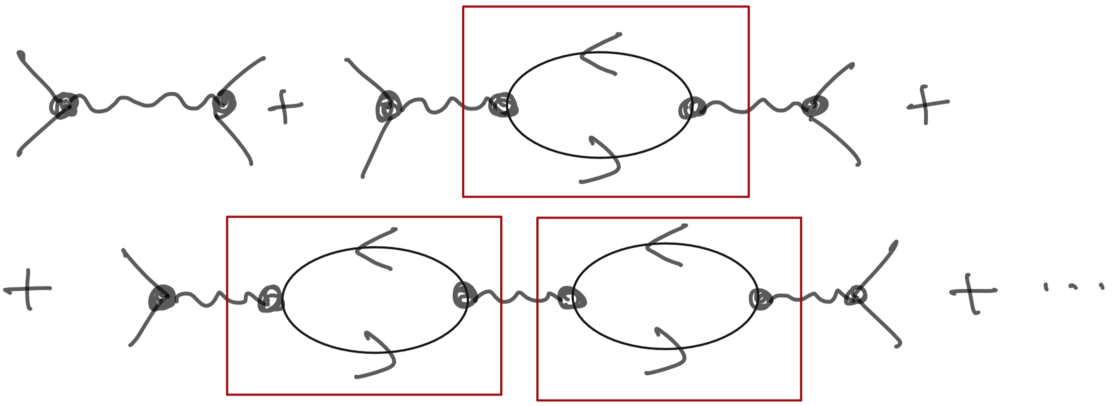
"Tr" summation over all spins of internal fermion/anti-fermion pair

Also this expression is uv-divergent as

$$\int d^4p \, \frac{p^2}{p^4} \sim \int^\infty d|p| \, |p| \longrightarrow \infty$$

As in the case of the electron self-energy we see that higher-order terms of the same structure exist

(478)



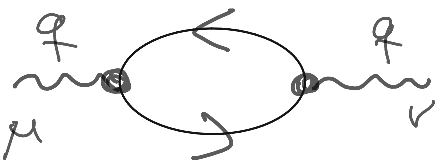
We will show later, that

(479) $$\boxed{\Pi^{\mu\nu}(q) = \left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) \Pi(q^2)}$$

where $\Pi(q^2)$ is a Lorentz scalar, so

$$q_\mu \Pi^{\mu\nu} = \Pi^{\mu\nu} q_\nu = 0$$

In a similar way as has been done for the self-energy, the higher-order contributions in (478) can be summed up (also here the summation is only a partial summation of the higher-order terms). For this we use

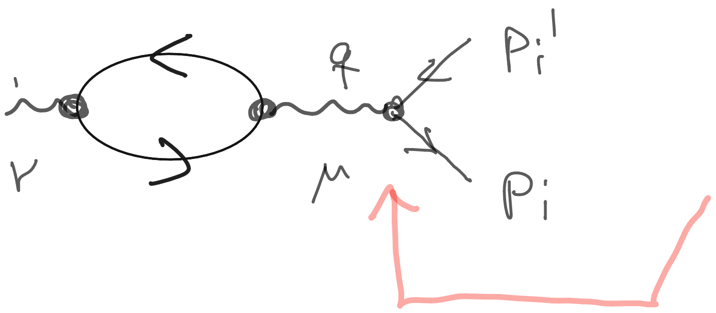


$$= \frac{i g^{\mu\lambda}}{-q^2 - i\varepsilon} \underbrace{\left(g_{\lambda\lambda'} q^2 - q_\lambda q_{\lambda'}\right) \Pi(q^2)}_{= \Pi_{\lambda\lambda'}(q^2)} \frac{i g^{\lambda'\nu}}{-q^2 - i\varepsilon}$$

(480)

$$= \frac{i}{-q^2-i\varepsilon}\left(g^{\mu\nu}q^2 - \underbrace{q^\mu q^\nu}_{\text{vanishes}}\right)\Pi(q^2)\frac{i}{-q^2-i\varepsilon} = \underbrace{\frac{-ig^{\mu\nu}}{-q^2-i\varepsilon}\Pi(q^2)}_{= -D^{\mu\nu}(q)\Pi(q^2)}$$

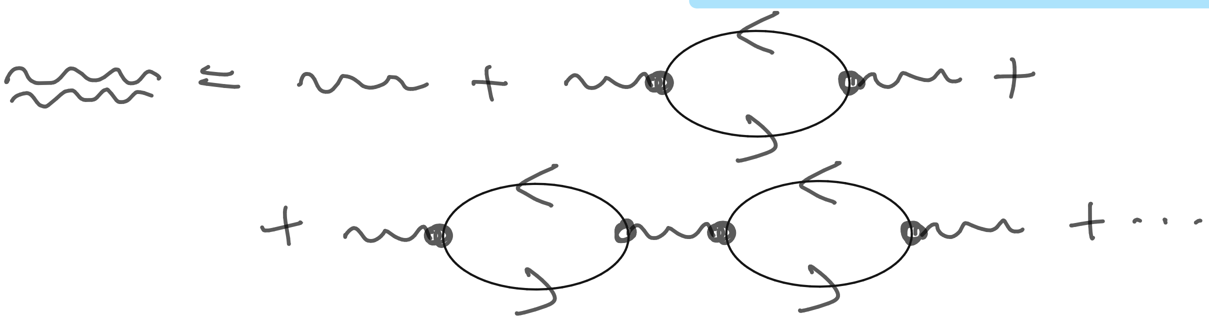
The red term vanishes when we carry out the contraction with connecting fermion lines



$$\bar{u}(p_i')\gamma^\mu v(p_i)q_\mu$$
$$= \bar{u}(p_i')\gamma^\mu v(p_i)(p_i - p_i')_\mu$$
$$= \bar{u}(p_i')(\underbrace{\cancel{p}_i - \cancel{p}_i'}_{\text{solution of Dirac eq}})v(p_i)$$
$$= \bar{u}(p_i')(m-m)v(p_i) = 0$$

from (478) we introduce the dressed photon propagator

(481)



$$D'^{\mu\nu}(q) = D^{\mu\nu}(q) + D^{\mu\nu}(q)(-\Pi(q^2))$$
$$+ D^{\mu\nu}(q)(-\Pi(q^2))(-\Pi(q^2)) + \dots$$

(482) $$\boxed{D'^{\mu\nu}(q) = D^{\mu\nu}(q)\frac{1}{1+\Pi(q^2)} = \frac{-g^{\mu\nu}}{q^2+i\varepsilon}\frac{1}{1+\Pi(q^2)}}$$

One recognizes that the pole of the photon propagator is still at $q^2=0$. So the interaction does not "create" an effective mass of the photon. In fact gauge invariance does not allow for the generation of an effective mass. This is more generally true. Also the W, Z bosons of the electroweak interaction and the gluons of QCD cannot have a mass. Here another mechanism, the Higgs mechanism, is needed.

Expanding $\Pi(q^2)$ around the pole $q^2=0$ yields

$$\Pi(q^2) = \Pi(0) + \bar{\Pi}(q^2)$$

with $\bar{\Pi}(0)=0$. So

(483) $$D'^{\mu\nu}(q^2) = \frac{g^{\mu\nu}}{-q^2-i\varepsilon} \, \frac{Z_3}{1+Z_3\bar{\Pi}(q^2)}$$

with

(484) $$\boxed{Z_3 \equiv \frac{1}{1+\Pi(0)}}$$ divergent

in the vicinity of the pole

(485) $$D'^{\mu\nu}(q^2) \xrightarrow[q^2\to 0]{} \frac{g^{\mu\nu}Z_3}{-q^2-i\varepsilon}$$

effect of vacuum polarization Π:

(i) no mass renormalization!
(gauge invariance)

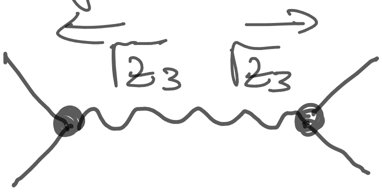
(ii) modification of the normalization of the photon propagator

Elimination of Z_3 by charge and wavefunction renormalization

- all internal photon lines connect to two vortices

(486) $$Z_3 \longrightarrow \sqrt{Z_3}\sqrt{Z_3}$$

thus we can absorb a factor $\sqrt{Z_3}$ into the charge at the end of an internal photon line



$$e_0 \longrightarrow \sqrt{Z_3}\sqrt{Z_2}\, e_0$$

- in external photon lines a factor $\sqrt{Z_3}$ remains, hower, so

rule (11) for Feynman diagrams

$\boxed{\text{multiply every external photon line with } \sqrt{Z_3}}$