(B) Feynman rules of QED

Generalizing the findings in (A) and in Sec. V.1–V.3 we can formulate the Feynman rules of QED

The scattering matrix element between asymptotic eigenstates of the free hamiltonian reads

$$S = -i(2\pi)^4 \, \delta^{(4)}\Big(\sum_{\text{final}} p_f - \sum_{\text{initial}} p_i\Big) \prod_{j=1}^{n} f_j \, \mathcal{M} \tag{444}$$

where the amplitude factors of the asymptotic states in the limit $L^3 \to \infty$ are

$$f_j = \frac{1}{\sqrt{(2\pi)^3 \, 2E_j}} \tag{445}$$

rules for calculating $\mathcal{M}$

(i) Draw all topologically different, connected diagrams consisting of interaction vertices, Dirac lines and photon lines. The external lines correspond to the incoming / outgoing particles / antiparticles / photons. The number of vertices corresponds to the order of perturbation theory. $e^2 = \alpha \left(= \frac{e^2}{\hbar c}\right) = \frac{1}{137}$

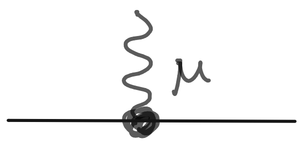
(ii) Each diagram corresponds to a mathematical expression according to the following rules. Add the expressions for all diagrams in a given order of perturbation theory

(1) Energy and momentum of external lines are given by those of the incoming / outgoing particles / photons. At each vertex there is 4-momentum conservation. In tree-like diagrams this fixes all 4-momenta in the diagram. In loop diagrams an integration

$$\int \frac{d^4k}{(2\pi)^4}$$

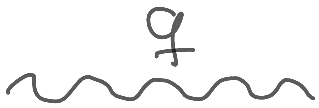
has to be carried out for each loop

(2) Every vertex at which a photon of polarization μ is emitted or absorbed by a fermion of (positive) charge e gets a matrix factor



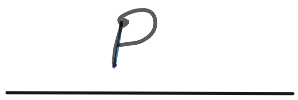
$$\boxed{-ie\gamma_\mu} \tag{446}$$

(3) Every internal photon line with 4-momentum q is described by the photon propagator



$$\boxed{\frac{-i g^{\mu\nu}}{q^2 + i\varepsilon}} \qquad q^2 = q_\mu q^\mu$$

(4) Every internal fermion line with 4-momentum p and Dirac indices α and β is described by the electron-positron propagator



$$\boxed{\frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}} \qquad p^2 = p_\mu p^\mu$$

(5) For incoming and outgoing fermion lines

- multiply from left with $\bar{u}(p,s)$ for outgoing fermion with 4-momentum p and spin s

- multiply from right with with $u(p,s)$ for incoming femions with 4-momentum p and spin s

- multiply from left with $\bar{v}(p,s)$ for outgoing anti-fermion with 4-momentum p and spin s

- multiply from right with $v(p,s)$ for incoming anti-fermion with 4-momentum p and spin s

(6) For incoming and outgoing photon lines

- multiply with $\vec{e}_m^{\,*}$ for outgoing photon of polarization m (formally $\vec{e}_\mu^{\,*}$)
- multiply with $\vec{e}_m$ for incoming photon of polarization m (formally $\vec{e}_\mu$)

(7) Symmetrize between all identical bosons in the incoming (or outgoing) states

(8) Provide a factor (-1) for each closed fermion loop

(9) Multiply loop diagrams with n identical boson lines with a factor $1/n!$

(C) Spin averages

Often one wants to calculate differential cross sections for unpolarized electrons / protons.

For this one needs to average over the spin degrees of freedom of incoming and outgoing fermions.

E.g. $$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

- For electron-proton scattering (see eq. 447)

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^2}{|\Delta q|^4} \sum_{s_f, s_i} \bar{u}(q_f, s_f)\gamma^\mu u(q_i, s_i)\, \bar{u}(q_i, s_i)\gamma^\nu u(q_f, s_f)$$

(447) $$\cdot \sum_{\sigma_f, \sigma_i} \bar{u}(p_f, \sigma_f)\gamma_\mu u(p_i, \sigma_i)\, \bar{u}(p_i, \sigma_i)\gamma_\nu u(p_f, \sigma_f)$$

Here we have used

$$\left[\bar{u}(q_f, s_f)\gamma^\mu u(q_i, s_i)\right]^* = \left[u^+(q_f, s_f)\gamma^0\gamma^\mu u(q_i, s_i)\right]^*$$

(448) $$= u^+(q_i s_i)\gamma^{\mu +}\gamma^0 u(q_f, s_f)$$

$$= u^+(q_i s_i)\gamma^0\gamma^\mu u(q_f, s_f) = \bar{u}(q_i, s_i)\gamma^\mu u(q_f, s_f)$$

You will show in the exercises that

$$\boxed{\begin{aligned}\Lambda_+(p) &= \frac{1}{2m}\sum_s u(p,s)\,\bar{u}(p,s) = \frac{m+\not{p}}{2m}\\ \Lambda_-(p) &= -\frac{1}{2m}\sum_s v(p,s)\,\bar{v}(p,s) = \frac{m-\not{p}}{2m}\end{aligned}} \quad (44f)$$

are projectors to positive (+) and negative (−) energy states

Thus in eq. (447)

$$\sum_{s_i,s_f} \bar{u}(q_f,s_f)\gamma^\mu \underbrace{u(q_i,s_i)}_{\text{4-row}}\,\underbrace{\bar{u}(q_i,s_i)}_{\text{4x4}}\gamma^\nu \underbrace{u(q_f,s_f)}_{\text{4-column}}$$

$$= \sum_{s_f} \bar{u}(q_f,s_f)\gamma^\mu(m_p+\not{q}_i)\gamma^\nu u(q_f,s_f)$$

$$= \sum_{s_f} \bar{u}_\alpha(q_f,s_f)\left[\gamma^\mu(m_p+\not{q}_i)\gamma^\nu\right]_{\alpha\beta} u_\beta(q_f,s_f)$$

$$= \sum_{s_f} u_\beta(q_f,s_f)\,\bar{u}_\alpha(q_f,s_f)\left[\gamma^\mu(m_p+\not{q}_i)\gamma^\nu\right]_{\alpha\beta}$$

$$= (m_p+\not{q}_f)_{\beta\alpha}\left[\gamma^\mu(m_p+\not{q}_i)\gamma^\nu\right]_{\alpha\beta}$$

$$= \mathrm{Tr}\left\{(m_p+\not{q}_f)\gamma^\mu(m_p+\not{q}_i)\gamma^\nu\right\}$$

$\Rightarrow$ $\boxed{\text{Spin sums} \;\hat{=}\; \text{trace of products of } \gamma\text{-matrices}}$

theorem 1: The trace of a product of an odd number of γ-matrices vanishes

(450)

proof: $\mathrm{Tr}\{\cancel{a}_1 \cancel{a}_2 \cdots \cancel{a}_n\} = \mathrm{Tr}\{\underbrace{\gamma^5 \gamma^5}_{=1} \cancel{a}_1 \cancel{a}_2 \cdots \cancel{a}_n\}$

$= (-1)^n \mathrm{Tr}\{\gamma^5 \cancel{a}_1 \cancel{a}_2 \cdots \cancel{a}_n \gamma^5\}$

n fold commutation

$= (-1)^n \mathrm{Tr}\{\cancel{a}_1 \cancel{a}_2 \cdots \cancel{a}_n\}$ $\square$

cyclic property of trace

Note: $\cancel{\gamma}^\mu = \cancel{e}_\mu$ $\quad (e_\mu)_\nu = \delta_{\mu\nu}$

theorem 2:

(451)

$n=0 \quad \mathrm{Tr}\{1\} = 4$

$n=2 \quad \mathrm{Tr}\{\cancel{a}\cancel{b}\} = 4 a_\mu b^\mu$

$n=4 \quad \mathrm{Tr}\{\cancel{a}\cancel{b}\cancel{c}\cancel{d}\} = 4\left(a_\mu b^\mu c_\nu d^\nu - a_\mu c^\mu b_\nu d^\nu + a_\mu d^\mu b_\nu c^\nu\right)$

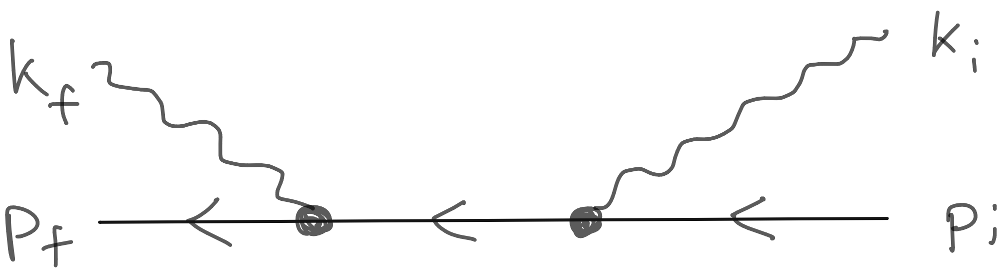
(D) Compton scattering

An important example is Compton scattering of a photon with an electron

$$\gamma + e \longrightarrow \gamma + e$$

The reduced scattering matrix element has only non-trivial contributions in 2nd order

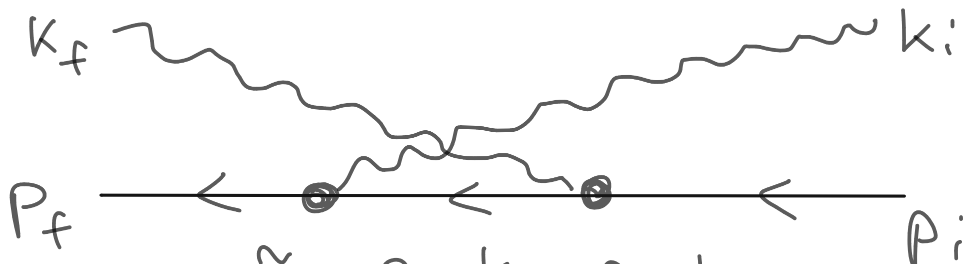
(a)



$$q = k_i + p_i = k_f + p_f$$

and

(b)



$$\tilde{q} = p_i - k_f = p_f - k_i$$

4-momentum conservation

$$p_i + k_i = p_f + k_f \qquad (457)$$

This gives

$$\mathcal{M} = -e^2\,\bar{u}(p_f)\Big[\gamma^\mu e_\mu^{f*}\,\frac{-i(m+\not{p}_i+\not{k}_i)}{m^2-(p_i+k_i)^2-i\varepsilon}\,\gamma^\nu e_\nu^i$$

(452)

$$+\gamma^\mu e_\mu^{i*}\,\frac{-i(m+\not{p}_i-\not{k}_f)}{m^2-(p_i-k_f)^2-i\varepsilon}\,\gamma^\nu e_\nu^f\Big]u(p_i)$$

remark: each of the two terms taken separately is not gauge invariant, the sum however is!

For the calculation of the scattering cross section one can use that the electron is at rest in the Lab-frame, so

(453) $p_i = (m, \vec{0})$ $\qquad e^\mu = (0, \vec{e})$ $\quad \vec{e}^{\,*} = \vec{e}$

electron 4-momentum $\qquad$ photon polarization

$$e_i k_i = e_f k_f = e_i p_i = e_f p_f = 0$$

transversality $\qquad$ see (453)

furthermore

$$k_i^2 = \underbrace{k_i^{0\,2}}_{\omega_i^2} - \vec{k}_i^{\,2} = k_f^2 = \underbrace{k_f^{0\,2}}_{\omega_f^2} - \vec{k}_f^{\,2} = 0$$

free photon dispersion relation

i.e.

$$(p_i+k_i)^2 = m^2 + 2m\omega_i$$

$$(p_i-k_f)^2 = m^2 - 2m\omega_f$$

So

$$\mathcal{M} = -e^2 \bar{u}(p_f)\Big\{ \cancel{e}^f \frac{m + \cancel{p}_i + \cancel{k}_i}{-2m\omega_i} \cancel{e}^i$$

(453)

$$+ \cancel{e}^i \frac{m + \cancel{p}_i - \cancel{k}_f}{-2m\omega_f} \cancel{e}^f \Big\} u(p_i)$$

Spin-averaging in incoming and outgoing fermions yields (in the lab-frame) the

Klein–Nishina formula of Compton scattering

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{lab} = \frac{\alpha^2}{4m^2}\left(\frac{k'}{k}\right)^2 \Big[4(\vec{e}_i \cdot \vec{e}_f)^2 + \underbrace{\frac{k}{k'} + \frac{k'}{k} - 2}_{\frac{(k-k')^2}{kk'}} \Big]}$$

(454)

where $k = |\vec{k}_i|$ $k' = |\vec{k}_f|$

and $\vec{e}_i$ and $\vec{e}_f$ are the corresponding photon polarizations.

For low-energy scattering $k' \ll m$ $k \approx k'$ and averaging over photon polarizations one finds

(455) $$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{unpol.} = \frac{\alpha^2}{m^2} \frac{1}{2}(1 + \cos^2\theta)}$$

(456) or $$\boxed{\sigma = \frac{8\pi}{3} \frac{\alpha^2}{m^2}}$$ Thomson scattering cross section