procedure:

(I) Draw all topologically distinct diagrams that correspond to a given scattering / decay process in the considered order of perturbation theory. The order of perturbation corresponds to the number of vertices

(II) Find mathematical expression for every diagram according to the Feynman rules and add expressions for all diagrams

(D) relativistic scattering in 2nd order of λ.

In the previous examples the incoming and outgoing particles were distinguishable.

Now: identical particles in incoming and/or outgoing channels

(4.10)
$$|\Psi_{in}\rangle = \hat{a}^{+}_{\vec{p}}\, \hat{a}^{+}_{\vec{q}}\, |0\rangle$$
$$|\Psi_{out}\rangle = \hat{a}^{+}_{\vec{p}'}\, \hat{a}^{+}_{\vec{q}'}\, |0\rangle$$

(4.11)
$$S = \frac{(-i)^2}{2}\lambda_1^2 \int d^4x_1 \int d^4x_2$$

$$\langle 0|\, \hat{d}_{1p'}\hat{d}_{1q'}\, T\left[:\hat{\Phi}_1^\dagger(x_1)\,\hat{\Phi}_1(x_1)\,\hat{\varphi}(x_1)::\hat{\Phi}_1^\dagger(x_2)\,\hat{\Phi}_1(x_2)\,\hat{\varphi}(x_2): \right] \hat{d}_{1p}^\dagger\, \hat{d}_{1q}^\dagger |0\rangle$$

A

A

B

B

C

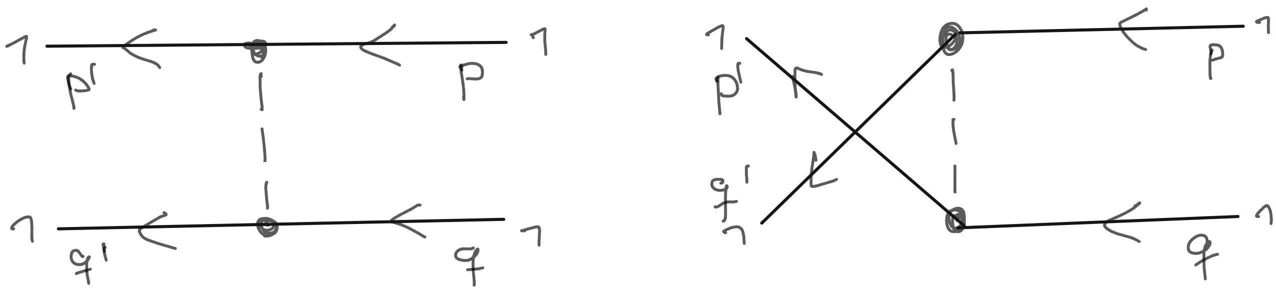
C

energy momentum conservation

(4.12)
$$p + q = p' + q'$$

(4.13)
$$\mathcal{M} \approx \frac{\lambda_1^2}{(p'-p)^2 - m_0^2 + i\varepsilon} + \frac{\lambda_1^2}{(q'-p)^2 - m_0^2 + i\varepsilon}$$

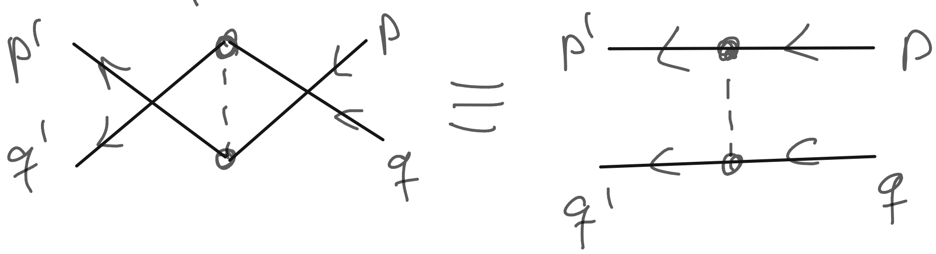
from A from B



rule 4: Symmetrize between identical bosons in the outgoing channels.

remark: rule 4 could alternatively also be formulated for incoming identical bosons, however not for both as

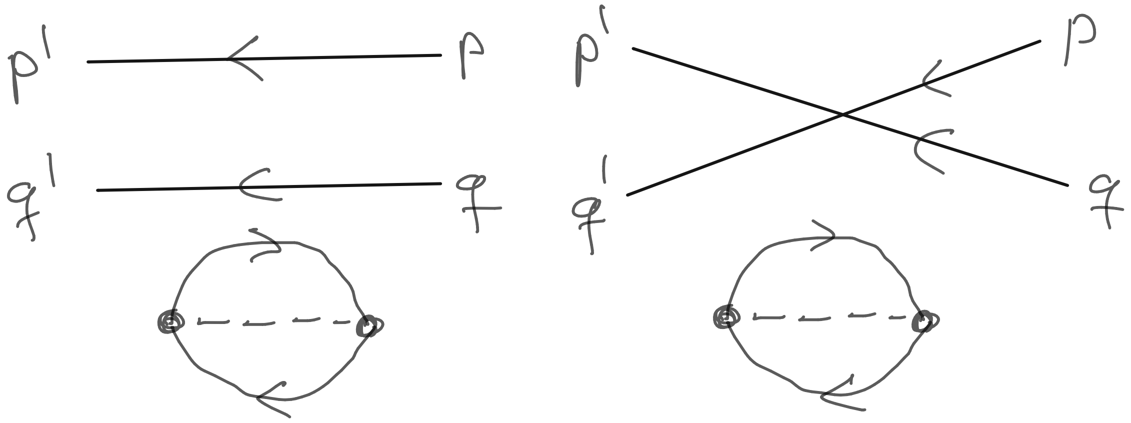
(414)



- how about terms from C ?

this corresponds to unconnected diagrams:

(415)



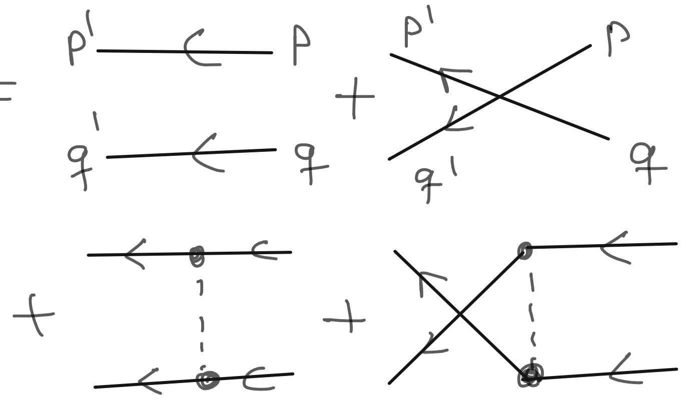
connected graph theorem

In a consistent perturbation theory of the S-matrix unconnected diagrams do not contribute, where "unconnected" diagrams are those that contain parts which do not connect to external lines.

idea for a proof: make use of normalization

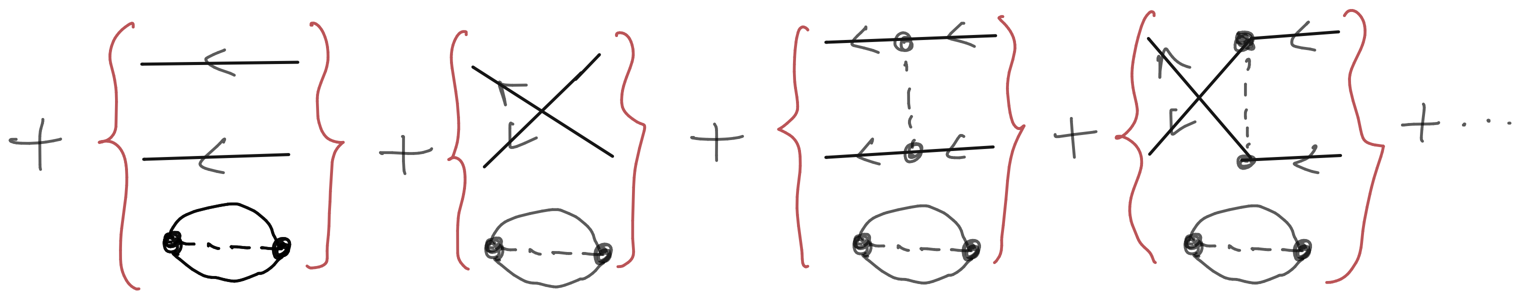
$$S_{\beta\alpha} = \langle\beta|\hat{U}_I(\infty,-\infty)|\alpha\rangle \rightarrow \frac{\langle\beta|\hat{U}_I(\infty,-\infty)|\alpha\rangle}{\langle 0|\hat{U}_I(\infty,-\infty)|0\rangle} \qquad (416)$$

$$\langle p' q'|\hat{U}_I|p\, q\rangle = \qquad (417)$$



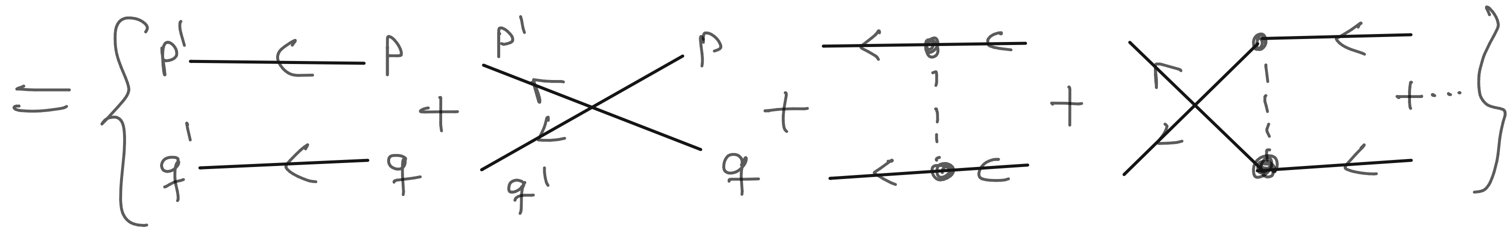
0'th order

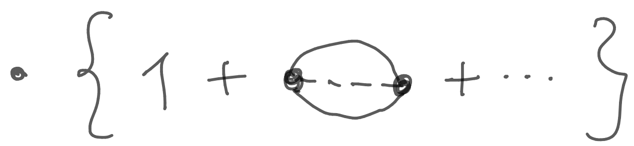
2nd order



2nd

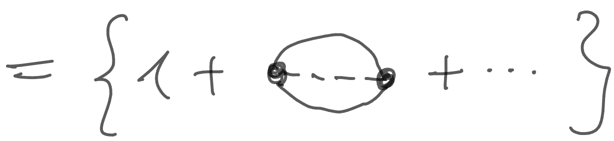
4th





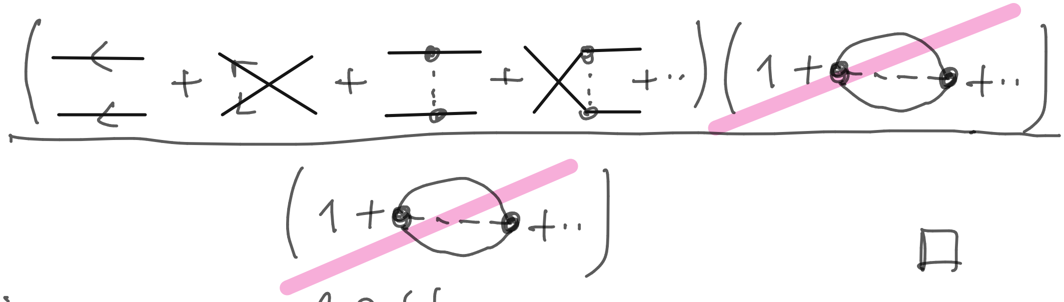
on the other hand

$$\langle 0|\hat{U}_I|0\rangle =$$



(418)

$$\frac{\langle p'q'|\hat{U}_I|pq\rangle}{\langle 0|\hat{U}_I|0\rangle} = \tag{419}$$



unconnected diagrams cancel out! $\square$

(E) loop diagrams in 2nd order of λ

Consider incoming and outgoing neutral particle in second order of λ

$$S = \frac{(-i)^2}{2}\frac{\lambda^2}{(3!)^2}\int d^4x_1 \int d^4x_2$$

2 possibilities

$$\langle 0|\hat{a}_k T\left[:\hat{\varphi}(x_1)\hat{\varphi}(x_1)\hat{\varphi}(x_1)::\hat{\varphi}(x_2)\hat{\varphi}(x_2)\hat{\varphi}(x_2):\right]\hat{a}^+_{k'}|0\rangle \tag{420}$$

3 possibilities

3 possibilities

3 possibilities

3 possibilities

$\Rightarrow$

$$S = \frac{(-i)^2}{2}\frac{\lambda^2}{2}\int d^4x_1 \int d^4x_2$$

$$\langle 0|\hat{a}_k(\infty)\hat{\varphi}(x_1)|0\rangle \left(\langle 0|T\,\hat{\varphi}(x_1)\hat{\varphi}(x_2)|0\rangle\right)^2 \langle 0|\hat{\varphi}(x_2)\hat{a}^+_{k'}(-\infty)|0\rangle \tag{421}$$

$$+ \text{"}x_1\text{"} \leftrightarrow \text{"}x_2\text{"}$$

so $$S = \frac{(-i)^2}{2}\frac{\lambda^2}{2}\frac{1}{\sqrt{2^2 E_k E_{k'} (2\pi)^6}}$$

(422) $$* \int d^4x_1 \int d^4x_2\; e^{-ik'x_2} e^{ikx_1} \left(\langle 0| T\hat{\varphi}(x_1)\hat{\varphi}(x_2)|0\rangle\right)^2$$
$$+ \text{"}x_1\text{"} \leftrightarrow \text{"}x_2\text{"}$$

We again introduce com and relative coordinates

$$R = \frac{1}{2}(x_1 + x_2) \qquad r = x_1 - x_2$$

(423) $$S = \frac{(-i)^2}{2}\frac{\lambda^2}{2}\frac{1}{\sqrt{2^2 E_k E_{k'} (2\pi)^6}}$$
$$* \int d^4r \int \underline{d^4R}\; e^{ik(\underline{R}+\frac{1}{2}r)} e^{-ik'(\underline{R}-\frac{1}{2}r)} \left(\langle 0| T\hat{\varphi}(x_1)\hat{\varphi}(x_2)|0\rangle\right)^2$$
$$+ \text{"}r\text{"} \leftrightarrow \text{"}-r\text{"}$$

$$= -\frac{\lambda^2}{4} f_k f_{k'} \underline{(2\pi)^4}\; \underline{\delta^{(4)}(k-k')}$$
$$\cdot \int d^4r\; e^{ikr} \left(\langle 0| T\hat{\varphi}(r)\hat{\varphi}(0)|0\rangle\right)^2 + \text{"}r\text{"} \leftrightarrow \text{"}-r\text{"}$$

Making use of the property of the Fourier trafo (convolution)

(424) $$\int d^4r\; e^{ikr} (h(r))^2 = \int \frac{d^4k''}{(2\pi)^4} \tilde{h}(k'') \tilde{h}(k-k'')$$

yields

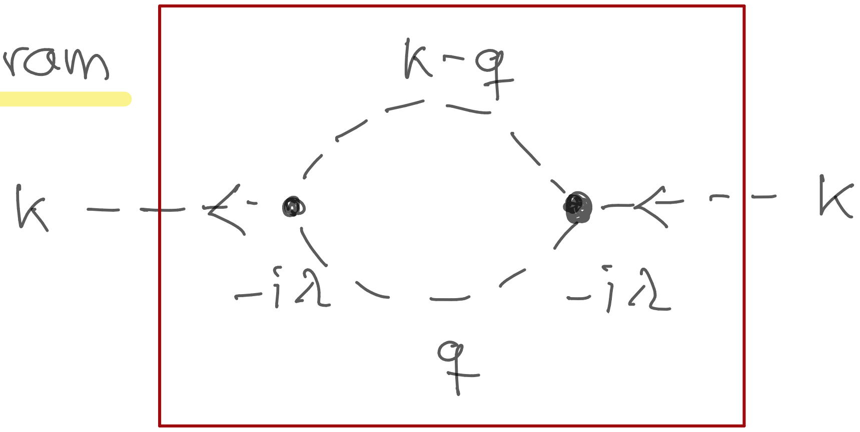
$$S = -\frac{\lambda^2}{2} f_k^2 \, \delta^{(4)}(k-k') \cdot (2\pi)^4 \int \frac{d^4k''}{(2\pi)^4} \frac{1}{k''^2 - m_0^2 + i\varepsilon} \, \frac{1}{(k-k'')^2 - m_0^2 + i\varepsilon} \tag{425}$$

The last expression corresponds to a reduced S-matrix element

$$\boxed{\mathcal{M} = i\,(-i\lambda)^2 \, \frac{1}{2!} \int \frac{d^4q}{(2\pi)^4} \, \frac{i}{q^2 - m_0^2 + i\varepsilon} \, \frac{i}{(k-q)^2 - m_0^2 + i\varepsilon}} \tag{426}$$

rule 7 rule 5

loop diagram



rule 5 : integrate over all 4-momenta which are not fixed by external 4-momenta by energy-momentum conservation. with weight

$$\int \frac{d^4q}{(2\pi)^4}$$

rule 7 : multiply loop diagrams with n internal lines of identical bosons with a factor $\frac{1}{n!}$

remark: loop diagrams yield in general divergent contributions and are the origin of mathematical problems of the theory. We will learn how to deal with these infinities using renormalization (see later)

V.4 Feynman rules of QED

We now want to establish the diagrammatic perturbation theory of QED. Starting point is the Hamiltonian in Coulomb gauge

$$\hat{H} = \hat{H}_{elm} + \hat{H}_D + \hat{H}_{int} \qquad (427)$$

$\mu = e, p$
electron, proton

with

$$\hat{H}_D = \sum_{\mu=e,p} \int d^3r\, \hat{\psi}_\mu^\dagger \left(-i\vec{\alpha}\cdot\vec{\nabla} + m_\mu \beta\right) \hat{\psi}_\mu$$

$$\hat{H}_{elm} = \frac{1}{2} \int d^3r \left(\hat{\vec{E}}_\perp^{\,2} + \hat{\vec{B}}^2\right) \qquad (428)$$

$$\hat{H}_{int} = \hat{H}_{int}^1 + \hat{H}_{int}^2$$

where

(429) $$\hat{H}^1_{int} = \frac{1}{4\pi}\int d^3r \int d^3r' \; \frac{\hat{j}_p^0(\vec{r},t)\,\hat{j}_e^0(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

instantaneous Coulomb interaction

(430) $$\hat{H}^2_{int} = -\int d^3r \left(\hat{\vec{j}}_p(\vec{r},t) + \hat{\vec{j}}_e(\vec{r},t)\right)\hat{\vec{A}}_\perp(\vec{r},t)$$

radiative interaction

The Coulomb self interaction $\sim \hat{j}_p^{0\,2}$ and $\sim \hat{j}_e^{0\,2}$ is assumed to be absorbed in $\hat{H}_D$.

(A) electron-proton scattering

2 Dirac fields electron $p_i \longrightarrow p_f$

proton $q_i \longrightarrow q_f$

For notational simplicity we now absorb the spin degree of freedom in p_i, i.e.

(431) $$\text{"}p_j\text{"} \;\hat{=}\; p_j, s_j$$

4 momentum spin

$\boxed{\hat{H}^1_{int}}$ yields contribution in 1st order

(432) $$I_1 \equiv -i\langle p_f, q_f | \int_{-\infty}^{\infty} d\tau\, \hat{H}^1_{int}(\tau) | p_i, q_i\rangle$$

with

$$\hat{j}^\mu_p = e\, \bar{\hat{\psi}}_p \gamma^\mu \hat{\psi}_p \qquad \text{proton}$$

(433)

$$\hat{j}^\mu_e = -e\, \bar{\hat{\psi}}_e \gamma^\mu \hat{\psi}_e \qquad \text{electron}$$

$\Rightarrow$

$$I_1 = \frac{ie^2}{4\pi}\int d^3r \int d^3r' \frac{\langle q_f | \bar{\hat{\psi}}_p(\vec{r})\gamma^0 \hat{\psi}_p(\vec{r}) | q_i\rangle\langle p_f | \bar{\hat{\psi}}_e(\vec{r}')\gamma^0 \hat{\psi}_e(\vec{r}') | p_i\rangle}{|\vec{r}-\vec{r}'|}$$

(434)

since

$$\hat{\psi}(x) = \sum_{\vec{p},s} \frac{e^{-iE_pt}}{\sqrt{2E_pL^3}}\left(u(\vec{p},s)\, e^{i\vec{p}\cdot\vec{r}}\, \hat{b}_{\vec{p},s} + v(\vec{p},s)e^{-i\vec{p}\cdot\vec{r}}\, \hat{d}^\dagger_{\vec{p},s}\right)$$

$$\hat{\psi}_e(x')|p_i\rangle = \frac{1}{\sqrt{2L^3}}\sum_{\vec{p},s}\frac{u(\vec{p},s)}{\sqrt{E_p}}\, e^{i\vec{p}\vec{r}'}\underbrace{\hat{b}_{\vec{p},s}|p_i\rangle}_{\delta_{\vec{p}\vec{p}_i}\delta_{ss_i}}$$

continuum limit $L\to\infty$ $\quad p_i = (\vec{p}_i, s_i)$

$$\hat{\psi}_e(x')|p_i\rangle = \frac{1}{\sqrt{2(2\pi)^3}}\frac{u(p_i)}{\sqrt{E_{p_i}}}\, e^{i\vec{p}_i\vec{r}'} e^{-iE_{p_i}t}$$

This yields for I_1

$$I_1 = \frac{ie^2}{4\pi} \frac{1}{\sqrt{16 E_{p_f} E_{p_i} E_{q_f} E_{q_i}}} \frac{1}{(2\pi)^6} \int_{-\infty}^{\infty} d\tau\, e^{i(E_{p_f}+E_{q_f}-E_{p_i}-E_{q_i})\tau}$$

(435)
$$\cdot \int d^3r \int d^3r' \cdot \frac{e^{-i(\vec{p}_f-\vec{p}_i)\vec{r}'} e^{-i(\vec{q}_f-\vec{q}_i)\vec{r}}}{|\vec{r}-\vec{r}'|}$$

$$\cdot \left[\bar{u}(q_f)\gamma^0 u(q_i)\right]\left[\bar{u}(p_f)\gamma^0 u(p_i)\right]$$

Introduce COM and relative coordinates

$$R = \frac{1}{2}(\vec{r}+\vec{r}') \qquad \rho = \vec{r}-\vec{r}'$$

- spatial integral $\quad (2\pi)^3 \delta^{(3)}((\vec{p}_f+\vec{q}_f)-(\vec{p}_i+\vec{q}_i)) \dfrac{4\pi}{|\Delta\vec{q}|^2}$

$$\Delta\vec{q} = \vec{q}_f - \vec{q}_i = \vec{p}_i - \vec{p}_f$$

- time integral $\quad 2\pi\, \delta(E_{p_f} + E_{q_f} - E_{p_i} - E_{q_i})$

$\Rightarrow$

(436)
$$\underline{\underline{I_1 = \frac{ie^2}{4\pi^2} \frac{\delta^{(4)}(q_f+p_f-q_i-p_i)}{\sqrt{16 E_f^p E_f^q E_i^p E_i^q}} \frac{1}{|\Delta\vec{q}|^2}}}$$

$$\underline{\underline{\cdot \left[\bar{u}(q_f)\gamma^0 u(q_i)\right] \cdot \left[\bar{u}(p_f)\gamma^0 u(p_i)\right]}}$$

$\boxed{H^2_{int}}$ contribution in 2nd order $\quad (qx = q_\mu x^\mu)$

$$I_2 = \frac{e^2}{2} \langle q_f p_f | \langle 0 | \int d^4x_1 \int d^4x_2$$

$$T\Big[\hat{A}^k_\perp(x_1) \hat{A}^l_\perp(x_2) :\bar{\hat{\psi}}_p(x_1)\gamma^k \hat{\psi}_p(x_1)::\bar{\hat{\psi}}_e(x_2)\gamma^l\hat{\psi}_e(x_2):$$

$$+ \quad \begin{matrix} x_1 \Leftrightarrow x_2 \\ k \Leftrightarrow l \end{matrix} \quad \Big] |0\rangle |q_i p_i\rangle$$

$$= \frac{e^2}{\sqrt{16 E^p_f E^p_i E^q_f E^q_i}} \frac{1}{(2\pi)^6} \int d^4x_1 \int d^4x_2 \, \langle 0 | T \hat{A}^k_\perp(x_1) \hat{A}^l_\perp(x_2) |0\rangle$$

(437) $$e^{i(q_f - q_i)x_1} e^{i(p_f - p_i)x_2}$$

$$\left[\bar{u}(q_f)\gamma^k u(q_i)\right] \left[\bar{u}(p_f)\gamma^l u(p_i)\right]$$

again COM and relative coordinates

$$X = \frac{1}{2}(x_1 + x_2) \qquad x = x_1 - x_2$$

$\Rightarrow$

$$I_2 = \frac{e^2}{\sqrt{16 E^p_f E^p_i E^q_f E^q_i}} \frac{1}{(2\pi)^6} \left[\bar{u}(q_f)\gamma^k u(q_i)\right] \left[\bar{u}(p_f)\gamma^l u(p_i)\right]$$

(438) $$\cdot \int d^4X \, e^{i(q_f - q_i + p_f - p_i)X} \rightarrow (2\pi)^4 \delta^{(4)}(q_f - q_i + p_f - p_i)$$

$$\cdot \int d^4x \, \langle 0 | T \hat{A}^k_\perp(x_1) \hat{A}^l_\perp(x_2) |0\rangle e^{i\Delta q x} \rightarrow \text{Ftrafo of photon propagator}$$

Thus $\qquad \Delta q = p_f - p_i = q_i - q_f \quad$ (4-momenta)

$$I_2 = \frac{ie^2}{4\pi^2}\,\frac{\delta^{(4)}(q_f+p_f-q_i-p_i)}{\sqrt{16E_f^pE_i^pE_f^qE_i^q}}\;\frac{\delta_{\kappa\ell}-\dfrac{\Delta q_\kappa \Delta q_\ell}{(\Delta\vec{q})^2}}{\Delta q^2+i\varepsilon}$$

$$\underline{\underline{\left[\bar{u}(q_f)\gamma^\kappa u(q_i)\right]\left[\bar{u}(p_f)\gamma^\ell u(p_i)\right]}} \tag{439}$$

We realize from (436) and (439) that both I_1 and I_2 are of 2nd order in α $(=e^2$ since $\hbar=c=1)$

We now show that

$$\boxed{S = I_1 + I_2 = \frac{\delta^{(4)}(q_f+p_f-q_i-p_i)}{\sqrt{16E_f^pE_i^pE_f^qE_i^q}}\,\frac{1}{(2\pi)^2}\,\mathcal{M}} \tag{440}$$

$$\boxed{\mathcal{M} = ie^2\left[u(q_f)\gamma_\mu u(q_i)\right]\left[\bar{u}(p_f)\gamma_\nu u(p_i)\right]\cdot\frac{-ig^{\mu\nu}}{\Delta q^2+i\varepsilon}} \tag{441}$$

full 4-dimensional propagator of elm. field (as in Lorentz gauge)

That $\mathcal{M}$ contains the full propagator is not surprising since physical observables should not depend on the gauge choice and we could have used the Lorentz gauge from the beginning!

<u>proof of (447)</u>

$$k=1,2,3 \rightarrow \mu=0,1,2,3$$
$$l=1,2,3 \rightarrow \nu=0,1,2,3$$

(i) $$\delta_{kl}-\frac{\Delta q_k \Delta q_l}{|\Delta\vec{q}|^2}=-g_{\mu\nu}-\frac{\Delta q_\mu \Delta q_\nu}{|\Delta\vec{q}|^2}$$
$$+\delta_{\mu 0}\frac{\Delta q_0 \Delta q_\mu}{|\Delta\vec{q}|^2}+\delta_{\nu 0}\frac{\Delta q_0 \Delta q_\nu}{|\Delta\vec{q}|^2}-\delta_{\mu 0}\delta_{\nu 0}\frac{\Delta q^2}{|\Delta\vec{q}|^2}$$

$$\Delta q^2=\Delta q_0^2-|\Delta\vec{q}|^2$$

(ii) $$\Delta q_\mu \bar{u}(p_f)\gamma^\mu u(p_i)=(p_f-p_i)_\mu \bar{u}(p_f)\gamma^\mu u(p_i)$$
$$=\bar{u}(p_f)(\not{p}_f-\not{p}_i)u(p_i)$$

Now $\bar{u}(p_f)$ and $u(p_i)$ are solutions of the free Dirac equation

$$(\gamma^\mu p_{i\mu}-m_e)u(p_i)=(\not{p}_i-m_e)u(p_i)=0$$
$$\bar{u}(p_f)(\gamma^\mu p_{f\mu}-m_e)=\bar{u}(p_f)(\not{p}_f-m_e)=0$$

Thus

$$I_2 = \frac{ie^2}{4\pi^2} \frac{\delta^{(4)}(p_f - p_i + q_f - q_i)}{\sqrt{16 E_f^p E_i^p E_f^q E_i^q}} \frac{1}{\Delta q^2 + i\varepsilon} \tag{442}$$

$$\left(-g_{\mu\nu} - \delta_{\mu 0}\delta_{\nu 0} \frac{\Delta q^2}{|\Delta \vec{q}|^2}\right) \bar{u}(q_f)\gamma^\mu u(q_i)\, \bar{u}(p_f)\gamma^\nu u(p_i)$$

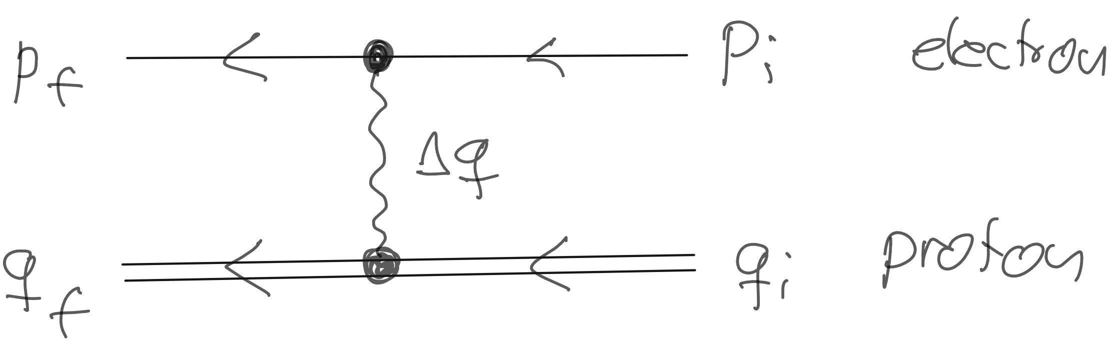
Furthermore

$$I_1 = \frac{ie^2}{4\pi^2} \frac{\delta^{(4)}(p_f - p_i + q_f - q_i)}{\sqrt{16 E_f^p E_i^p E_f^q E_i^q}} \frac{1}{\Delta q^2 + i\varepsilon} \tag{443}$$

$$\delta_{\mu 0}\delta_{\nu 0} \frac{\Delta q^2}{|\Delta \vec{q}|^2} \bar{u}(q_f)\gamma^\mu u(q_i)\, \bar{u}(p_f)\gamma^\nu u(p_i)$$

This proves eqs (440) and (441) $\square$

diagrammatic representation of $\mathcal{M}$ (441)



4-momentum transfer: Δq