In order to see the matrix structure of S in eq. (382) it is convenient to use the following equivalence

(385) $$\boxed{S(k) = \frac{1}{\not{k} - m + i\varepsilon} = \frac{\not{k} + m}{k_\mu k^\mu - m^2 + i\varepsilon}}$$

(keeping terms in leading order of ε)

Note that $(\not{k} + m)_{\alpha\beta} = \gamma^\mu_{\alpha\beta} k_\mu + m\delta_{\alpha\beta}$

V.3 Diagrammatic perturbation theory: an introduction for the ϕ^3 theory

We will now introduce a systematic approach to calculate S-matrix elements in perturbation theory. For this we consider the so-called ϕ^3 theory of a neutral and charged Klein-Gordon field

(386) $$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{int}$$

neutral KG field $\qquad \hat{\varphi} = \hat{\varphi}^{+}$

(387) $$\mathcal{L}_0 = :\frac{1}{2}\frac{\partial\hat{\varphi}}{\partial x^\mu}\frac{\partial\hat{\varphi}}{\partial x_\mu} - m_0^2\hat{\varphi}^2:$$

charged KG fields $\qquad \hat{\Phi}_{1,2} \qquad j=1,2$

(388) $$\mathcal{L}_{1/2} = :\frac{1}{2}\frac{\partial\hat{\Phi}_j^{+}}{\partial x^\mu}\frac{\partial\hat{\Phi}_j}{\partial x_\mu} - m_j^2\hat{\Phi}_j^{+}\hat{\Phi}_j:$$

interaction

$$\mathcal{L}_{int} = -\lambda_1 :\hat{\Phi}_1^{+}(x)\hat{\Phi}_1(x)\hat{\varphi}(x):$$

(389) $$-\lambda_2 :\hat{\Phi}_2^{+}(x)\hat{\Phi}_2(x)\hat{\varphi}(x):$$

$$-\frac{\lambda}{3!}:\hat{\varphi}^3(x): \quad = -\mathcal{H}_{int}$$

normal order $\Rightarrow \quad \langle 0|\mathcal{H}_{int}|0\rangle = 0$

mode decomposition:

(390) $$\hat{\Phi}_j(x) = \sum_p\left(\hat{d}_{jp}\Phi_{jp}^{(+)}(x) + \hat{c}_{jp}^{+}\Phi_{j-p}^{(-)}(x)\right)$$

(391) $$\hat{\varphi}(x) = \sum_k\left(\hat{a}_k\varphi_k^{(+)}(x) + \hat{a}_k^{+}\varphi_{-k}^{(-)}(x)\right)$$

all wave functions are K.-G.-wavefunctions

$$(392) \qquad \varphi_k^{(\pm)} = \frac{1}{\sqrt{2E_k L^3}} e^{i(\vec{k}\cdot\vec{r} \mp E_k t)}$$

similar $\Phi_p^{(\pm)}$

The time evolution operator in the interaction picture up to 2nd order

$$\hat{U}_I = 1 - i \int d^4x \, \hat{\mathcal{H}}_I(x) + \frac{(-i)^2}{2} \int d^4x_1 \int d^4x_2 \, T\left[\hat{\mathcal{H}}_I(x_1) \hat{\mathcal{H}}_I(x_2)\right] + \dots$$

Let us now calculate matrix elements in perturbation theory

(A) relativistic decay of neutral particle in a pair of charged ones

neutral particle with 4-momentum k

$$|k\rangle = \hat{a}_k^+ |0\rangle$$

$$\downarrow$$

particle-antiparticle pair of type 1 with 4-momenta $p, \bar{p}'$

$$|p, \bar{p}'\rangle = \hat{d}_{1p}^+ \hat{c}_{1p'}^+ |0\rangle$$

(overbar to indicate antiparticle)

$$\boxed{S = \langle p, \bar{p}' | \hat{U}_I(\infty, -\infty) | k \rangle = ?}$$

There is a contribution in 1st order perturbation theory in λ_1

$$S = -i \langle 0 | \hat{d}_{1p} \hat{c}_{1p'} \int d^4x \, \lambda_1 : \hat{\Phi}_1^+(x) \hat{\Phi}_1(x) \varphi(x) : \hat{a}_k^+ | 0 \rangle$$

This can be written as time-ordered product

$d_{1p} \to d_{1p}(+\infty)$ $\qquad$ $a_k \to a_k(-\infty)$

$$S = -i\lambda_1 \int d^4x \, \langle 0 | T \, \hat{d}_{1p} \hat{c}_{1p'} : \hat{\Phi}_1^+(x) \hat{\Phi}_1(x) \hat{\varphi}(x) : \hat{a}_k^+ | 0 \rangle \tag{393}$$

non-vanishing contractions

all fields are bosons; only "external" contractions

$$S = -i\lambda_1 \int d^4x \, \langle 0 | T \left(\hat{d}_{1p}(\infty) \hat{\Phi}_1^+(x) \right) | 0 \rangle$$

$$\langle 0 | T \left(\hat{c}_{1p'}(\infty) \hat{\Phi}_1(x) \right) | 0 \rangle \tag{394}$$

$$\langle 0 | T \left(\hat{\varphi}(x) \hat{a}_k^+(-\infty) \right) | 0 \rangle$$

Now

$$\langle 0 | T \left[\hat{d}_{1p}(\infty) \hat{\Phi}_1^+(x) \right] | 0 \rangle = \langle 0 | \hat{d}_{1p}(\infty) \hat{\Phi}_1^+(x) | 0 \rangle$$

$$= \Phi_{1p}^{(+)*}(x) = \frac{1}{\sqrt{2E_p L^3}} e^{ipx} \qquad px = E_p t - \vec{p}\cdot\vec{r}$$

$$\langle 0 | T[\hat{c}_{1p'}(\infty)\hat{\Phi}_1(x)]|0\rangle = \langle 0|\hat{c}_{1p'}(\infty)\hat{\Phi}_1(x)|0\rangle$$

$$= \Phi_{1-p'}^{(-)}(x) = \frac{1}{\sqrt{2E_{p'}L^3}} e^{ip'x}$$

and

$$\langle 0| T[\hat{\varphi}(x)\hat{a}_k^{+}(-\infty)]|0\rangle = \langle 0|\hat{\varphi}(x)\hat{a}_k^{+}(-\infty)|0\rangle$$

$$= \varphi_k^{(+)}(x) = \frac{1}{\sqrt{2E_k L^3}} e^{-ikx}$$

Thus in 1st order

$$S = -i\lambda_1 \int d^4x\, \Phi_{1p}^{(+)*}(x)\, \Phi_{1-p'}^{(-)}(x)\, \varphi_k^{(+)}(x)$$

(395)
$$= -i\lambda_1 \int d^4x \frac{e^{i(p+p'-k)x}}{\sqrt{2^3 E_p E_{p'} E_k L^9}}$$

in the limit $V_4 \to \infty$ we find

(396)
$$\boxed{S = -i(2\pi)^4 \delta^{(4)}(p+p'-k) \prod_j f_j \mathcal{M}}$$

energy-momentum conservation between incoming and outgoing plane wave states

$$\boxed{f_j = \frac{1}{\sqrt{(2\pi)^3 2E_j}}}$$

amplitude factor for incoming and outgoing waves

$\boxed{\mathcal{M}}$ reduced scattering matrix element in the given order of perturbation theory

Here it is simply $\mathcal{M} = \lambda_1$

Eq. (395) is the general form of the S-matrix for decay processes!

(B) relativistic scattering in 1+1 order perturbation

charged particle of type 1 $\quad p$

+ charged particle of type 2 $\quad q$

$$|p,q\rangle = \hat{a}^+_{1p}\, \hat{a}^+_{2q}\, |0\rangle$$

$$\downarrow$$

charged particle of type 1 p'
+ charged particle of type 2 q'

$$|p', q'\rangle = \hat{a}^{+}_{1p'} \hat{a}^{+}_{2q'} |0\rangle$$

Here we need 2nd order of $\hat{\mathcal{H}}_I$

(two terms of the same type)

$$S = \frac{(-i)^2}{2} \, 2 \, \lambda_1 \lambda_2 \int d^4x_1 \int d^4x_2 \; * \qquad (397)$$

$$\langle 0| \hat{a}_{1p'} \hat{a}_{2q'} T\left[:\hat{\Phi}^{+}_1(x_1) \hat{\Phi}_1(x_1) \hat{\varphi}(x_1) :: \hat{\Phi}^{+}_2(x_2) \hat{\Phi}_2(x_2) \hat{\varphi}(x_2) : \right] \hat{a}^{+}_{1p} \hat{a}^{+}_{2q} |0\rangle$$

$\Rightarrow$ non-vanishing contractions

"external" contractions, see (A)

"internal" contraction

$$S = -\lambda_1 \lambda_2 \int d^4x_1 \int d^4x_2 \; \underline{\Phi^{(+)*}_{1p'}(x_1) \Phi^{(+)}_{1p}(x_1)} \; \underline{\Phi^{(+)*}_{2q'}(x_2) \Phi^{(+)}_{2q}(x_2)}$$

(398) $$\underline{\langle 0| T\left[\hat{\varphi}(x_1) \hat{\varphi}(x_2)\right] |0\rangle}$$

Let us introduce relative and center-of-mass coordinates

(399) $$R = \frac{1}{2}(x_1 + x_2) \qquad r = x_1 - x_2$$

$$(x = t, \vec{r})$$

then

$$\int d^4x_1 \int d^4x_2 = \int d^4R \int d^4r$$

Due to translational invariance of the Hamiltonian and the free KG theory

$$\langle 0|T[\hat{\varphi}(x_1)\hat{\varphi}(x_2)]|0\rangle = \langle 0|T[\hat{\varphi}(r)\hat{\varphi}(0)]|0\rangle$$

$\Rightarrow$

$$S = -\lambda_1\lambda_2 \int d^4R \int d^4r \frac{e^{i\{(p'-p)(R+\frac{1}{2}r)+(q'-q)(R-\frac{1}{2}r)\}}}{\sqrt{2^4(2\pi)^{3\cdot 4} E_p E_{p'} E_q E_{q'}}} \tag{400}$$

$$\cdot \langle 0|T[\hat{\varphi}(r)\hat{\varphi}(0)]|0\rangle$$

the $\int d^4R$ integral can be carried out to yield

$$S = -\lambda_1\lambda_2 \frac{(2\pi)^4 \delta^{(4)}(p'+q'-p-q)}{\sqrt{(2(2\pi^3))^4 E_p E_q E_{p'} E_{q'}}} \tag{401}$$

$$\int d^4r\, e^{i(q'-q)r} \langle 0|T[\hat{\varphi}(r)\hat{\varphi}(0)]|0\rangle$$

This has a similar structure as the expression for the decay

$$\boxed{S = -i(2\pi)^4\delta^{(4)}(p'+q'-p-q)\prod_j f_j\, \mathcal{M}} \tag{402}$$

now with $\quad \Delta q r = (q' - q) r = (q'_\mu - q_\mu) r^\mu$

$$\boxed{\mathcal{M} = -i\lambda_1\lambda_2 \int d^4r\, e^{i\Delta q r} \langle 0|T[\hat{\varphi}(r)\hat{\varphi}(0)]|0\rangle} \tag{403}$$

$\Delta q = q' - q = p - p'$ momentum transfer

- energy-momentum conservation between incoming and outgoing particles
- $f_j = \dfrac{1}{\sqrt{2(2\pi)^3 E_j}}$ amplitude factors of all incoming and outgoing waves
- reduced scattering matrix element

In the problem course you will show that

$$-i\int d^4x\, e^{i\Delta q x} \langle 0|T[\hat{\varphi}(r)\hat{\varphi}(0)]|0\rangle$$

$$= \frac{1}{\Delta q^2 - m_0^2 + i\varepsilon} \tag{404}$$

i.e.

$$\mathcal{M} = \frac{-\lambda_1\lambda_2}{m_0^2 - (q'-q)^2 - i\varepsilon} \tag{405}$$

(C) Feynman diagrams: introduction

We have seen in sections (A) and (B) important simple examples for the calculation of S-matrix elements where in- and outgoing states are plane waves i.e. eigenstates of 4-momentum

S matrix element for relativistic decay / scattering with plane-wave, free-particle asymptotic states have the form

(406) $$S = -i(2\pi)^4 \delta^{(4)}\Big(\sum_{in} p - \sum_{out} p'\Big) \prod_j f_j \mathcal{M}$$

where $\delta^{(4)}$ accounts for energy-momentum conservation between asymptotic states, $f_j = 1/\sqrt{(2\pi)^3 2E_j}$ are amplitude factors of all incoming and outgoing waves and $\mathcal{M}$ is the reduced matrix element

For more complex processes in higher order perturbation theory the calculation of $\mathcal{M}$ can be complicated $\Rightarrow$ need set of rules for systematic calculation of $\mathcal{M}$

$\Rightarrow$ $\boxed{\text{Feynman rules}}$

We will also find a graphical representation of these rules

$$\boxed{\text{Feynman diagrams}}$$

Feynman diagrams

relevant example of interaction

$$\mathcal{H}_{int} = g :\hat{\bar{\psi}}\hat{\psi}\hat{\phi}: \qquad (407)$$

ψ - Dirac field $\qquad\qquad \phi$ - Bose field

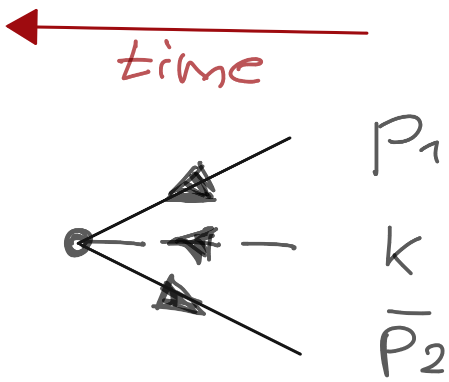
- each operator $\hat{\bar{\psi}}, \hat{\psi}, \hat{\phi}$ creates or annihilates a particle or anti-particle

- application of $\mathcal{H}_{int}$ with n interacting fields

$n-\ell$ particles in input
ℓ particles in output $\quad (0 \le \ell \le n)$

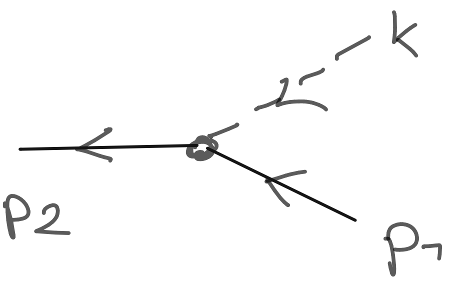
for $\hat{\bar{\psi}}\hat{\psi}\hat{\phi}$

$\boxed{\ell = 0}$

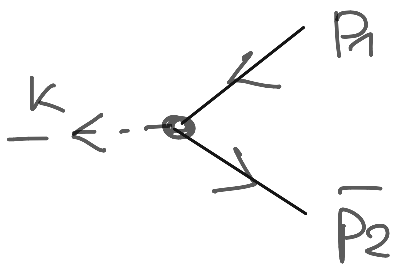


3 particle annihilation

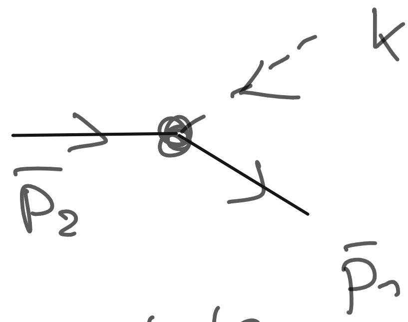
$\boxed{l=1}$



particle absorption

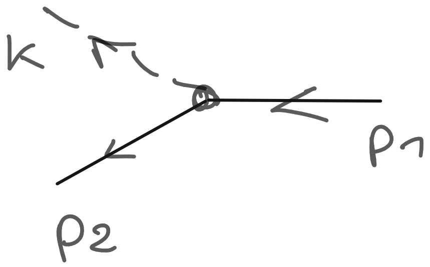


particle-anti-particle annihilation

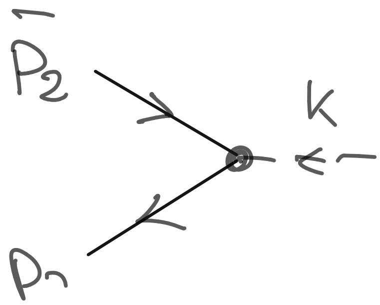


particle absorption

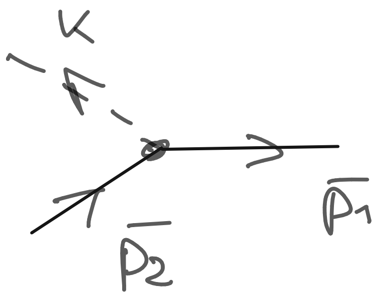
$\boxed{l=2}$



particle emission

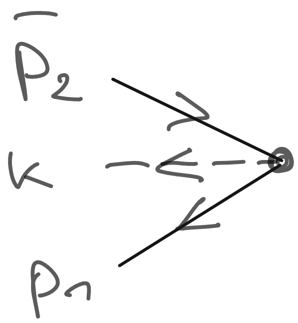


particle-anti-particle creation



particle emission

$\boxed{l=3}$



3-particle creation

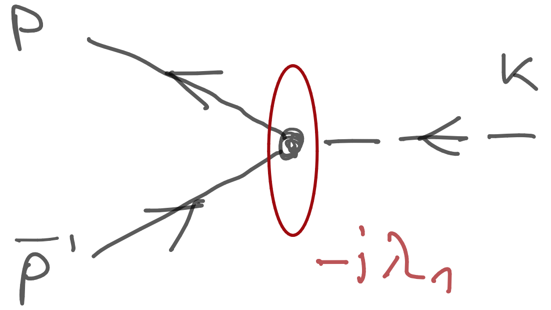
——<——	p	$\psi, \bar{\psi}$	particle with momentum p
——>——	$\bar{p}$	$\psi, \bar{\psi}$	antiparticle with momentum $\bar{p}$
- -<- -	k	ϕ	boson field with momentum k

example: $\tilde{\phi}^3$ theory

(i) relativistic decay in 1st order λ_1

outgoing particle with p

outgoing anti-particle with $\bar{p}'$



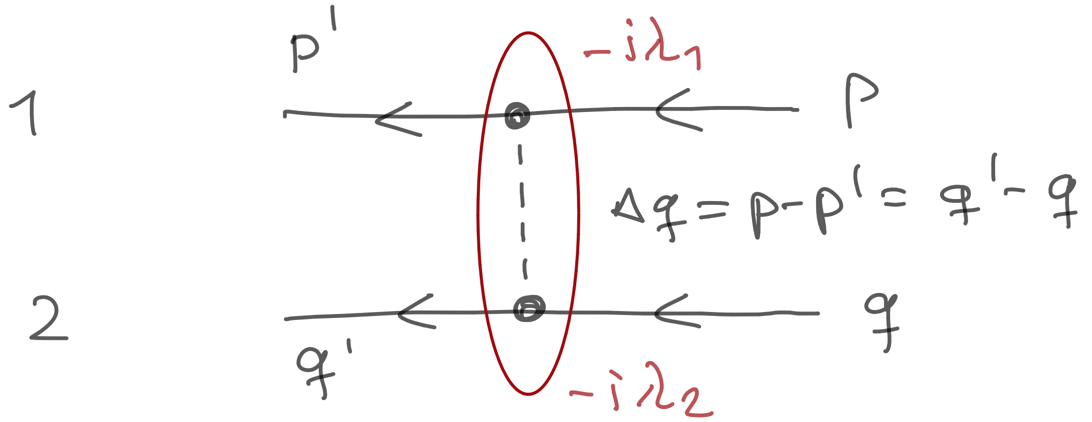
incoming $\tilde{\phi}$ particle with 4-momentum k

$-i\lambda_1$ Vertex from interaction Hamiltonian $-i\mathcal{H}_{1I}$

$$\boxed{\mathcal{M} = i(-i\lambda_1) = \lambda_1} \qquad (4.08)$$

rule 0

(ii) relativistic scattering in 1+1 st order $\lambda_1\lambda_2$



 propagator of $\tilde{\phi}$ with 4-moment $\Delta q = p - p' = q' - q$

$-i\lambda_1$, $-i\lambda_2$ vertices from interaction Hamiltonian $\mathcal{H}_{1I}$ and $\mathcal{H}_{2I}$

$$\boxed{\mathcal{M} = \underbrace{i}_{\text{rule 0}} \underbrace{(-i\lambda_1)(-i\lambda_2)}_{\text{rule 1}} \underbrace{\frac{i}{\Delta q^2 - m_0^2 + i\varepsilon}}_{\text{rule 2}}} \tag{409}$$

Feynman rules for calculation of $\mathcal{M}$

rule 0	• a factor of "i"
rule 1	• a factor of "$-i\lambda_k$" for every vertex • at each vertex 4-momentum is conserved
rule 2	• for every internal line a factor corresponding to the propagator of the field with 4-momentum Δq, determined by 4-momentum conservation at the vertices

remark: these are the rules for a theory $\mathcal{F}\,4\phi$. Most of them are universal, some need modifications. There will be more rules procedure:

(I) Draw all topologically distinct diagrams that correspond to a given scattering / decay process in the considered order of perturbation theory. The order of perturbation corresponds to the number of vertices

(II) Find mathematical expression for every diagram according to the Feynman rules and add expressions for all diagrams

(D) relativistic scattering in 2nd order of λ_1

In the previous examples the incoming and outgoing particles were distinguishable.

Now: identical particles in incoming and/or outgoing channels

(410)
$$|\Psi_{in}\rangle = \hat{a}^+_{1p}\,\hat{a}^+_{1q}\,|0\rangle$$
$$|\Psi_{out}\rangle = \hat{a}^+_{1p'}\,\hat{a}^+_{1q'}\,|0\rangle$$