V. QED with interactions: perturbation theory

In the following we will discuss quantum electrodynamics as the most important example of an interacting quantum field theory. For applications to atomic physics we need at least two sorts of fermions electrons and protons, which are coupled to the electromagnetic field. In fully covariant notation the Lagrangian reads

$$\mathcal{L}_{QED} = \mathcal{L}_e + \mathcal{L}_p - \frac{1}{4} \hat{F}^{\mu\nu} \hat{F}_{\mu\nu} - \hat{j}_\mu \hat{A}^\mu \tag{345}$$

where

$$\mathcal{L}_e = \frac{1}{2} \hat{\bar{\psi}}_e \left(i\gamma^\mu \overleftrightarrow{\partial}_\mu - m_e \right) \hat{\psi}_e$$

$$\mathcal{L}_p = \frac{1}{2} \hat{\bar{\psi}}_p \left(i\gamma^\mu \overleftrightarrow{\partial}_\mu - m_p \right) \hat{\psi}_p \tag{346}$$

$$\hat{j}^\mu = e \left(\hat{\bar{\psi}}_p \gamma^\mu \hat{\psi}_p - \hat{\bar{\psi}}_e \gamma^\mu \hat{\psi}_e \right)$$

opposite charges

In Coulomb gauge we eliminate A^0 and find

$$\mathcal{L}_{QED} = \mathcal{L}_e + \mathcal{L}_p + \frac{1}{2}\left(\hat{\vec{E}}_\perp^{\,2} + \hat{\vec{B}}^{\,2}\right) + \vec{j}\cdot\vec{A}_\perp$$

(347)
$$- \frac{1}{2}\int \frac{d^3r'}{4\pi} \, \frac{\rho(\vec{r},t)\,\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

This gives a Hamiltonian

$$\hat{H}_{QED} = \hat{H}_p^0 + \hat{H}_e^0 + \hat{H}_{elm}^0 + \frac{1}{4\pi}\int d^3r \int d^3r' \, \frac{\hat{\rho}_p(\vec{r},t)\,\hat{\rho}_e(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

(348)
$$- \int d^3r \left(\hat{\vec{j}}_p(\vec{r},t) + \hat{\vec{j}}_e(\vec{r},t)\right)\cdot\hat{\vec{A}}_\perp(\vec{r},t)$$

where the Coulomb-selfinteraction terms $\sim \hat{\rho}_e\hat{\rho}_e$, $\hat{\rho}_p\hat{\rho}_p$ were dropped

(349)
$$\hat{H}_e^0 = \int d^3r \; \hat{\psi}_e^{\dagger}(\vec{r},t)\left[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m_e\right]\hat{\psi}_e(\vec{r},t)$$

and similarly for $\hat{H}_p^0$

(350)
$$\hat{H}_{elm}^0 = \frac{1}{2}\int d^3r \left[\hat{\vec{E}}_\perp^{\,2}(\vec{r},t) + \hat{\vec{B}}^{\,2}(\vec{r},t)\right]$$

While $\hat{H}_e^0$, $\hat{H}_p^0$ and $\hat{H}_{em}^0$ are bilinear in the field operators and can be treated easily, $\hat{H}_{\text{Coulomb}}$ and $\hat{H}_{\text{rad}}$ no longer are

(351) $$\boxed{\hat{H}_{\text{Coulomb}} = \frac{1}{4\pi}\int d^3r \int d^3r' \frac{\hat{J}_e^0(\vec{r},t)\,\hat{J}_p^0(\vec{r}',t)}{|\vec{r}-\vec{r}'|}}$$

(352) $$\boxed{\hat{H}_{\text{rad}} = -\int d^3r \left(\hat{\vec{J}}_e(\vec{r},t) + \hat{\vec{J}}_p(\vec{r},t)\right)\cdot\hat{\vec{A}}_\perp(\vec{r},t)}$$

with

(353) $$\hat{J}_e^0 = -e\,\hat{\bar{\psi}}_e\gamma^0\hat{\psi}_e = -e\,\hat{\psi}_e^\dagger\hat{\psi}_e = \hat{\rho}_e$$

(354) $$\hat{J}_p^0 = e\,\hat{\bar{\psi}}_p\gamma^0\hat{\psi}_p = e\,\hat{\psi}_p^\dagger\hat{\psi}_p = \hat{\rho}_p$$

These do not allow for an analytic treatment and we need a systematic approximation method to calculate S-matrix elements!

We will see that such a method naturally emerges in QED as perturbation theory with the fine-structure constant $\alpha \simeq \frac{1}{137}$ as small parameter.

V.1 Wick theorem

In the following we want to calculate transition probability amplitudes (S-matrix elements) perturbatively, i.e.

$$\langle \beta | \hat{U}_I(\infty, -\infty) | \alpha \rangle$$

$$(355) \qquad = \langle \beta | T \exp\left\{ -i \int_{-\infty}^{\infty} d\tau \, \hat{H}_I(\tau) \right\} | \alpha \rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\tau_1 \cdots \int_{-\infty}^{\infty} d\tau_n \, (-i)^n \langle \beta | T \hat{H}_I(\tau_1) \hat{H}_I(\tau_2) \cdots \hat{H}_I(\tau_n) | \alpha \rangle$$

where the interaction Hamiltonian contains e.g. $H_{Coulomb}$ and H_{rad}. Thus we need to evaluate terms like

$$\langle \beta | T \hat{H}_I(\tau_1) \cdots \hat{H}_I(\tau_n) | \alpha \rangle$$

We will be mostly concerned about scattering problems, where $|\alpha\rangle$ and $|\beta\rangle$ are incoming or outgoing plane waves, i.e.

$$|\alpha\rangle = \hat{a}^+_{\vec{q}} |0\rangle$$

and generalizations thereof

Thus it will be sufficient to calculate vacuum expectation values of the type

$$\langle 0| T \hat{a}_{\vec{q}'}(\infty) \hat{H}_I(\tau_1) \cdots \hat{H}_I(\tau_n) \hat{a}^\dagger_{\vec{q}}(-\infty) |0\rangle$$

left-most under T ordering

right-most under T ordering

As typically $\hat{H}_I \sim \hat{\psi}^\dagger \hat{A} \hat{\psi}$ we need to find a systematic way to calculate expressions like

$$\langle 0| T \hat{a}(\tau_1) \hat{a}(\tau_2) \hat{c}^\dagger(\tau_2) \cdots |0\rangle = ?$$

For this we can use the Wick theorem

Def: Time-ordered product of two field operators ($x_0 \equiv t_0$)

$$T\left[\hat{\phi}(x)\hat{\phi}^{(+)}(y)\right] = \begin{cases} \hat{\phi}(x)\hat{\phi}^{(+)}(y) & x_0 > y_0 \\ \eta\, \hat{\phi}^{(+)}(y)\hat{\phi}(x) & x_0 < y_0 \end{cases}$$

$\eta = +1$ Bose $\qquad \eta = -1$ Fermi

(356)

For many operators one finds

$$T\left[\hat{A}(x)\hat{B}(y)\hat{C}(z)\right] = \begin{cases} \hat{A}(x)\hat{B}(y)\hat{C}(z) & x_0 > y_0 > z_0 \\ \eta\,\hat{A}(x)\hat{C}(z)\hat{B}(y) & x_0 > z_0 > y_0 \\ \eta'\,\hat{B}(y)\hat{A}(x)\hat{C}(z) & y_0 > x_0 > z_0 \\ \text{etc.} & \end{cases} \tag{357}$$

$\eta, \eta' = (-1)^P$

P: # of permutations of fermion operators required to reach time-ordered configuration

Def: Normal-ordered product of two field operators

$$:\hat{\phi}(x)\hat{\phi}^{(+)}(y): \equiv \hat{\phi}(x)\hat{\phi}^{(+)}(y) - \langle 0|\hat{\phi}(x)\hat{\phi}^{(+)}(y)|0\rangle \tag{358}$$

Bose fields:

$$:\hat{a}\hat{a}^{+}: = \hat{a}\hat{a}^{+} - \langle 0|\hat{a}\hat{a}^{+}|0\rangle = \hat{a}\hat{a}^{+} - 1 = \hat{a}^{+}\hat{a} \tag{359}$$

Fermi fields:

$$:\hat{c}\hat{c}^{+}: = \hat{c}\hat{c}^{+} - \langle 0|\hat{c}\hat{c}^{+}|0\rangle = -\hat{c}^{+}\hat{c} + 1 - 1 = -\hat{c}^{+}\hat{c} \tag{360}$$

$\Rightarrow$ Normal ordering means moving creation operators to the left and annihilation operators to the right

Normal ordering of products of multiple annihilation and creation operators:

- permute all creation operators to the left and all annihilation operators to the right
- provide a prefactor $(-1)^p$ where p is the number of times fermionic operators need to be exchanged to arrive at the normal-ordered expression

example:

$$:\hat{a}\hat{a}^{+}\hat{a}^{+}\hat{a}\hat{a}: \; = \; \hat{a}^{+2}\hat{a}^{3}$$

$$:\hat{c}_1\hat{c}_1^{+}\hat{c}_2^{+}: \; = \hat{c}_1^{+}\hat{c}_2^{+}\hat{c}_1 = -\hat{c}_2^{+}\hat{c}_1^{+}\hat{c}_1$$

both normal ordered

Def: Contraction of two operators $\hat{A}(x)$ and $\hat{B}(x)$ which are linear functions of annihilation and creation operators

(361) $$\hat{A}(x)\hat{B}(y) \equiv \langle 0|\, T\hat{A}(x)\hat{B}(y)\,|0\rangle$$

let: $$\hat{\phi}(x) = \hat{\phi}^{(-)}(x) + \hat{\phi}^{(+)}(x)$$

contains only creation operators

contains only annihilation operators

Obviously:

$$\overbracket{\hat\phi^{(+)}\hat\phi^{(+)}} \equiv \overbracket{\hat\phi^{(-)}\hat\phi^{(-)}} \equiv 0$$

(362)
$$\overbracket{\hat\phi^{(\pm)}\hat\psi^{(\pm)}} \equiv 0$$

but also
$$\overbracket{\hat\phi^{(\pm)}\hat\psi^{(\mp)}} \equiv 0$$

Wick theorem

The time-ordered (T) product of operators linear in annihilation and creation operators equals the sum of all possible normal-ordered products with all possible pairwise contractions including the term without contractions. In addition each term gets a prefactor ± 1 depending on the number of pairwise permutations of fermion operators needed to obtain the normal ordered expressions.

i.e. $$T[\hat\phi_1\hat\phi_2\cdots\hat\phi_n] = :\hat\phi_1\hat\phi_2\cdots\hat\phi_n:$$

$$+ :\overbracket{\hat\phi_1\hat\phi_2}\hat\phi_3\cdots\hat\phi_n: \pm :\overbracket{\hat\phi_1\hat\phi_2\hat\phi_3}\hat\phi_4\cdots\hat\phi_n: \pm \cdots$$

$$\pm :\overbracket{\hat\phi_1\hat\phi_2}\,\overbracket{\hat\phi_3\hat\phi_4}\hat\phi_5\cdots\hat\phi_n: \pm \cdots$$

<u>remark:</u> in many-body theory there exist other more general formulations of Wick's theorem

<u>idea for a proof:</u>

(i) fix time arguments and put operators in time order

(ii) replace successively neighboring operators according to

$$\hat{\phi}_1 \hat{\phi}_2 = :\hat{\phi}_1 \hat{\phi}_2: - \langle 0 | \hat{\phi}_1 \hat{\phi}_2 | 0 \rangle$$

since operators were initially put into T-order

$$\langle 0 | \hat{\phi}_1 \hat{\phi}_2 | 0 \rangle = \langle 0 | T \hat{\phi}_1 \hat{\phi}_2 | 0 \rangle = \overbracket{\hat{\phi}_1 \hat{\phi}_2}$$

<u>important consequence of Wick's theorem</u>

(363) $$\langle 0 | T (\hat{\phi}_1 \cdots \hat{\phi}_{2n+1}) | 0 \rangle = 0$$ odd number

$$\langle 0 | T (\hat{\phi}_1 \cdots \hat{\phi}_{2n}) | 0 \rangle = \overbracket{\phi_1 \phi_2}\, \overbracket{\phi_3 \phi_4} \cdots \overbracket{\phi_{2n-1} \phi_{2n}}$$

$$+ \eta\, \overbracket{\phi_1 \phi_3}\, \overbracket{\phi_2 \phi_4} \cdots \overbracket{\phi_{2n-1} \phi_{2n}} + \cdots$$ even number

(364) $\hat{=}$ all possible complete pairwise contractions

$\Rightarrow$ The vacuum expectation value of a T-ordered product of $2n$ operators (which are linear in annihilation and creation operators) can be written as a sum of products of all possible n pairwise contractions

number of terms: $\dfrac{(2n)!}{2^n n!} = (2n-1)(2n-3)$

$\Rightarrow$ all can be reduced to $\phi(x)\phi(y)$ for non-interacting fields!

V.2 Contractions and propagators

We will see now that the contractions resulting from the application of Wick's theorem have a direct physical interpretation in terms of Greensfunction (propagators) of the free field equations.

(A) Elm. field in Coulomb gauge

fondamental fields:

$$\vec{A}_\perp(\vec{r},t) \qquad \vec{E}_\perp(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}_\perp(\vec{r},t)$$

The field equations in the presence of an external (prescribed) current read

(365) $$\Box \vec{A}_\perp(\vec{r},t) = \vec{j}_\perp(\vec{r},t) = \int d^3r' \, \vec{j}(\vec{r}',t)\, \overleftrightarrow{\delta}^{(3)}_\perp(\vec{r}-\vec{r}')$$

We can define a Greensfunction or propagator of this equation via

(366) $$\boxed{\Box D_c^{jk}(\vec{r},t;\vec{r}',t') = -\delta_\perp^{jk}(\vec{r}-\vec{r}')\,\delta(t-t')}$$

We will now proof that

(367) $$\boxed{D_c^{jk}(\vec{r},t;\vec{r}',t') = -i\langle 0|\, T\, \hat{A}_\perp^j(\vec{r},t)\hat{A}_\perp^k(\vec{r}',t')\,|0\rangle}$$

where $\hat{A}_\perp^j$ is the free (i.e. non-interacting) transverse vector potential

proof: $\hat{A}_\perp \to \hat{A}$ notation $x = \vec{r},t$
$x' = \vec{r}',t'$

$$\partial_t \langle 0| T\, \hat{A}^k(x)\hat{A}^\ell(x')|0\rangle =$$

$$= \partial_t \Big\{ \Theta(t-t')\langle 0|\hat{A}^k(x)\hat{A}^\ell(x')|0\rangle$$

$$+\Theta(t'-t)\langle 0|\hat{A}^\ell(x')\hat{A}^k(x)|0\rangle \Big\}$$

$$= \delta(t-t')\underbrace{\langle 0|[\hat{A}^k(x),\hat{A}^\ell(x')]|0\rangle}_{=0}$$

$$+\Theta(t-t')\langle 0|\partial_t\hat{A}^k(x)\,\hat{A}^\ell(x')|0\rangle$$

$$+\Theta(t'-t)\langle 0|\hat{A}^\ell(x')\,\partial_t\hat{A}^k(x)|0\rangle$$

$\Rightarrow$

$$\partial_t^2 \langle 0| T\hat{A}^k(x)\hat{A}^\ell(x')|0\rangle =$$

$$= \delta(t-t')\langle 0|[\underbrace{\partial_t\hat{A}^k(x)}_{=-\hat{E}^k(x)},\hat{A}^\ell(x')]|0\rangle$$

$$+\Theta(t-t')\langle 0|\partial_t^2\hat{A}^k(x)\,\hat{A}^\ell(x')|0\rangle$$

$$+\Theta(t'-t)\langle 0|\hat{A}^\ell(x')\,\partial_t^2\hat{A}^k(x)|0\rangle$$

$$= -\delta(t-t')\,\delta_\perp^{\ell k}(\vec{r}-\vec{r}\,')$$

$$+\Theta(t-t')\langle 0|\Delta\hat{A}^k(x)\hat{A}^\ell(x')|0\rangle$$

$$+\Theta(t'-t)\langle 0|\hat{A}^\ell(x')\Delta\hat{A}^k(x)|0\rangle$$

$\Rightarrow$

$$\boxed{\begin{aligned}&\Box_x \langle 0| T \hat{A}^k(x) \hat{A}^\ell(x')|0\rangle = \\ &= (\partial_t^2 - \Delta_{\vec{r}}) \langle 0| T \hat{A}^k(x) \hat{A}^\ell(x')|0\rangle = -i\,\delta_\perp^{k\ell}(\vec{r}-\vec{r}\,')\delta(t-t')\end{aligned}}$$

(368) $\square$

Often we consider infinitely extended space, i.e. without boundary conditions. Then

$$D_c^{k\ell}(\vec{r},t;\vec{r}\,',t') = D_c^{k\ell}(\vec{r}-\vec{r}\,';t-t')$$

Then often the Fourier representation in $\vec{k},\omega$ space is used

Def:

$$X(k) = X(k_0,\vec{k}) = X(\omega,\vec{k}) =$$

(369)
$$\equiv \int_{V_\infty} d^3r \int_{-\infty}^{\infty} dt\; e^{ik_\mu x^\mu} X(\vec{r},t)$$

where $k_\mu x^\mu = \omega t - \vec{k}\cdot\vec{r}$

In $(\vec{k},\omega)$-space eq. (368) turns into an algebraic equation

(370)
$$-(\omega^2 - \vec{k}^2) D_c^{k\ell}(\omega,\vec{k}) = -\int d^3r\, e^{-i\vec{k}\vec{r}} \delta_\perp^{k\ell}(\vec{r})$$

we had $\delta_\perp^{k\ell}(\vec{r}) = \delta_{k\ell}\,\delta(\vec{r}) - \delta_\parallel^{k\ell}(\vec{r})$

$$= \delta_{k\ell}\,\delta(\vec{r}) - \frac{1}{4\pi}\left(\nabla \circ \nabla \frac{1}{r}\right)^{k\ell}$$

F-trafo yields

(371) $$\int d^3r\, e^{-i\vec{k}\vec{r}}\,\delta_\perp^{k\ell}(\vec{r}) = \delta_{k\ell} - \frac{k_k k_\ell}{|\vec{k}|^2}$$

Thus

(372) $$\boxed{D_c^{ij}(k) = D_{c\,ij}(k) = \frac{\delta_{ij} - \frac{k_i k_j}{|k|^2}}{\omega^2 - |\vec{k}|^2 + i\varepsilon}}$$

Here the "$+\varepsilon$" was introduced to move the pole at $\omega = |\vec{k}|$ into the complex plane. The positive sign corresponds to the choice of the retarded Greensfunction.

(B) Elm. field in Lorentz gauge

If one wants to keep explicit Lorentz invariance in the formulation of QED one needs to add a gauge fixing term to the Lagrangian (Gupta-Bleuler)

(373) $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\partial_\mu A^\mu \partial_\nu A^\nu$$

Then there is a canonical momentum to A^0

$$\pi^0 = -\frac{1}{\alpha}\partial_\mu A^\mu$$

The Lorentz gauge is eventually realized by restricting the allowed quantum states such that

$$\partial_\mu \hat{A}^\mu |\psi\rangle = 0 \qquad \text{Gupta-Bleuler}$$

The Euler-Lagrange eqs. following from (373) are then

(374) $$\Box A^\nu - \left(1-\frac{1}{\alpha}\right)\partial^\nu\partial_\mu A^\mu = 0$$

For the choice of $\alpha=1$ (Fermi- or Feynmann gauge) one has

$$\Box A^\nu = 0$$

and in an external, i.e. prescribed 4-current

(375) $$\Box A^\nu = j^\nu$$

Here the fundamental commutation relations are then

(376) $$\boxed{\left[\hat{A}^\mu(\vec{r},t), \hat{\pi}^\nu(\vec{r}',t)\right] = i g^{\mu\nu}\delta(t-t')}$$

i.e. for $\mu=\nu=0$ the sign is reversed

Following the same procedure as in (A) one finds that

(377) $$\boxed{D_{\mu\nu}(x-x') = -i \langle 0| T \hat{A}_\mu(x) \hat{A}_\nu(x') |0\rangle}$$

is Greensfunction or propagator of the wave-equation

(378) $$\boxed{\Box_x D_{\mu\nu}(x-x') = g_{\mu\nu}\, \delta^{(4)}(x-x')}$$

F-trafo $x \to k = (\omega, \vec{k})$ yields

(379) $$\boxed{D^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{\omega^2 - \vec{k}^2 + i\varepsilon} = \frac{-g^{\mu\nu}}{k_\mu k^\mu + i\varepsilon}}$$

(C) Dirac field

Now lets consider the Dirac equation in an external i.e. prescribed elm. field

(380) $$(i\gamma^\mu \partial_\mu - m)\psi = -e\gamma^\mu A_\mu \psi$$

We define the Greensfunction or propagator as the (4×4) matrix $S(x,x')$

$$\boxed{\left(i\gamma^{\mu}\partial_{\mu}-m\right)S(x,x') = \mathbb{1}_{4\times4}\,\delta^{(4)}(x-x')}$$

(381)

Fourier-trafo for $S(x,x') = S(x-x')$ yields

$$\left(\gamma^{\mu}k_{\mu}-m\right)S(k) = \mathbb{1}$$

$\Rightarrow$

(382)
$$\boxed{S(k) = \frac{1}{\gamma^{\mu}k_{\mu}-m+i\varepsilon} = \frac{1}{\not{k}-m+i\varepsilon}}$$

where we have added again a term "$+i\varepsilon$" to move the pole into the complex plane. The "+" sign corresponds to the retarded propagator

Notation:

(383)
$$\boxed{\not{a} \equiv \gamma^{\mu}a_{\mu}}$$

Feynman dagger

In the problem course you will show that

(384)
$$\boxed{S(x,x') = -i\langle 0|T\,\hat{\psi}(x)\,\hat{\bar{\psi}}(x')\,|0\rangle}$$

Note that S is a 4×4 matrix, and

$$S_{\mu\nu}(x,x') = -i\langle 0|T\,\hat{\psi}_{\mu}(x)\,\hat{\bar{\psi}}_{\nu}(x')\,|0\rangle$$

In order to see the matrix structure of S in eq. (382) it is convenient to use the following equivalence

(385) $$\boxed{S(k) = \frac{1}{\not{k} - m + i\varepsilon} = \frac{\not{k} + m}{k_\mu k^\mu - m^2 + i\varepsilon}}$$

(keeping terms in leading order of ε)

Note that $(\not{k} + m)_{\alpha\beta} = \gamma^\mu_{\alpha\beta} k_\mu + m\,\delta_{\alpha\beta}$

V.3 Diagrammatic perturbation theory: an introduction for the ϕ^3 theory

We will now introduce a systematic approach to calculate S-matrix elements in perturbation theory. For this we consider the so-called ϕ^3 theory of a neutral and charged Klein-Gordon field

(386) $$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{int}$$