IV.3 Charge conjugation, spatial inversion & time-reversal, PCT theorem

(A) Parity and spatial inversion

spatial inversion $\quad \hat{P}: \vec{r} \longrightarrow -\vec{r} \qquad (302)$

$$\hat{P}^{+} = \hat{P}^{-1} \qquad \hat{P}^{2} = \mathbb{1}$$

action onto scalar field ϕ

(303) $$\boxed{\hat{P}\,\hat{\phi}(\vec{r},t)\,\hat{P}^{-1} = \eta^{P}_{\phi}\,\hat{\phi}(-\vec{r},t)}$$

since $\hat{P}^{2} = 1 \quad \Rightarrow \quad \boxed{\eta^{P}_{\phi} = \pm 1} \qquad (304)$

scalar fields are even or odd under inversion

action onto 4-vector field $\hat{A}^{\mu}$

(305) $$\boxed{\hat{P}\,\hat{A}^{\mu}(\vec{r},t)\,\hat{P}^{-1} = \Lambda^{\mu}{}_{\nu}\,\hat{A}^{\nu}(-\vec{r},t)}$$

(306) $$\boxed{\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}$$

4-vector fields are even under inversion

e.g. $\qquad j = 1,2,3$

(307) $\hat{P}\, \hat{E}^j(\vec{r},t)\, \hat{P}^{-1} = -\hat{E}^j(-\vec{r},t)$ as $\hat{E}^j = -\partial_t \hat{A}^j$

axial vector

(308) $\hat{P}\, \hat{B}^j(\vec{r},t)\, \hat{P}^{-1} = \hat{B}^j(-\vec{r},t)$ as $\hat{B}^j = (\vec{\nabla}\times\vec{A})^j$

polar vector

action onto bi-spinor field $\hat{\psi}^\mu$?

We would like that in the Dirac equation all terms transform in the same way under inversion! (similar arguments as for Lorentz-invariance)

$$\hat{P}\hat{x}^\mu\hat{P}^{-1} = \Lambda^\mu{}_\nu \hat{x}^\nu \qquad \hat{P}\,\partial_\mu\,\hat{P}^{-1} = \Lambda^\mu{}_\nu \partial_\nu$$

$$\Rightarrow \hat{P}(i\gamma^\mu\partial_\mu - m)\psi\hat{P}^{-1} = 0$$

$\Rightarrow$

(309)
$$\boxed{\hat{P}\gamma^\mu\hat{P}^{-1} = \Lambda^\mu{}_\nu\gamma^\nu}$$

i.e.

(310)
$$\hat{P}\gamma^0\hat{P}^{-1} = \gamma^0 \qquad \hat{P}\gamma^j\hat{P}^{-1} = -\gamma^j$$

i.e. $\hat{P} = e^{i\phi}\gamma^0 \qquad \hat{P}^2 = 1$

(311) $\Rightarrow$ $\boxed{e^{i\phi} = \pm 1 = \eta^P_\psi}$ spinor fields are even or odd

(312) $\boxed{\hat{P}\,\hat{\psi}(\vec{r},t)\,\hat{P}^{-1} = \eta^P_\psi\,\gamma^0\,\hat{\psi}(-\vec{r},t)}$

What is the action of $\hat{P}$ onto creation and annihilation operators?

$\boxed{\hat{\phi}}$ Klein-Gordon scalar field

$$\hat{P}\,\hat{\phi}(\vec{r},t)\,\hat{P}^{-1} = \eta^P_\phi\,\hat{\phi}(-\vec{r},t)$$

$$= \eta^P_\phi \sum_{\vec{k}} \frac{1}{\sqrt{2E_k L^3}}\left(\hat{a}_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{r}} e^{-iEt} + \hat{c}^{+}_{\vec{k}}\, e^{i\vec{k}\vec{r}} e^{iEt}\right)$$

$$= \eta^P_\phi \sum_{k} \frac{1}{\sqrt{2E_k L^3}}\left(\hat{a}_{-\vec{k}}\, e^{-ik_\mu x^\mu} + \hat{c}^{+}_{-\vec{k}}\, e^{ik_\mu x^\mu}\right)$$

thus

(313)
$$\boxed{\begin{aligned}\hat{P}\,\hat{a}_{\vec{k}}\,\hat{P}^{-1} &= \eta^P_\phi\,\hat{a}_{-\vec{k}}\\ \hat{P}\,\hat{c}_{\vec{k}}\,\hat{P}^{-1} &= \eta^P_\phi\,\hat{c}_{-\vec{k}}\end{aligned}}$$

parity changes momentum $\vec{k} \to -\vec{k}$

$\boxed{\hat{A}^{\mu}_{\perp}}$ transverse elm. field

$$\hat{P}\,\vec{\hat{A}}_{\perp}(\vec{r},t)\,\hat{P}^{+} = -\vec{\hat{A}}_{\perp}(-\vec{r},t)$$

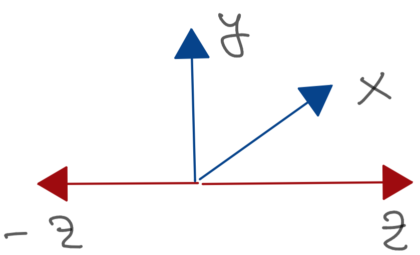
spatial components

circular polarization (similar to $\hat{\phi}$:)

$$= -\sum_{k,\sigma=\pm} \frac{1}{\sqrt{2\omega L^3}} \left(\vec{e}^{\,\sigma}_{-k}\, \hat{a}_{-\vec{k},\sigma}\, e^{-ik_\mu x^\mu} + \vec{e}^{\,\sigma *}_{-k}\, \hat{a}^{+}_{-k,\sigma}\, e^{ik_\mu x^\mu} \right)$$

now

$$\hat{P}\,\vec{e}^{\,\pm}_{z}\,\hat{P}^{-1} = \hat{P}\frac{1}{\sqrt{2}}(\vec{e}_x \pm i\vec{e}_y)\hat{P}^{-1}$$



$$= -\frac{1}{\sqrt{2}}(\vec{e}_x \pm i\vec{e}_y)$$

$$= \vec{e}^{\,\mp}_{-z}$$

direction of propagation

$\Longrightarrow$

(314)
$$\boxed{\hat{P}\,\hat{a}_{\vec{k},\pm}\,\hat{P}^{-1} = -\hat{a}_{-\vec{k},\mp}}$$

parity changes $\vec{k} \to -\vec{k}$ and circular polarizations

$\boxed{\hat{\psi}}$ Dirac field

$$\hat{P}\,\hat{\psi}(\vec{r},t)\,\hat{P}^{-1} = \eta^{P}_{\psi}\,\gamma^0\,\hat{\psi}(-\vec{r},t)$$

$$= \eta^{P}_{\psi}\,\gamma^0 \sum_{k,s} \frac{1}{\sqrt{2EL^3}} \Big(\hat{c}_{-\vec{k},s}\, u(-\vec{k},s)\, e^{-ik_\mu x^\mu} + \hat{d}^{+}_{-\vec{k},s}\, v(-\vec{k},s)\, e^{ik_\mu x^\mu} \Big)$$

Dirac spinors u and v obey

$$\gamma^0 u(-\vec{k}, s) = u(\vec{k}, s) \tag{315}$$
$$\gamma^0 v(-\vec{k}, s) = -v(\vec{k}, s)$$

Thus

$$\boxed{\begin{aligned} \hat{P}\,\hat{c}_{\vec{k},s}\,\hat{P}^{-1} &= \eta^P_{\psi}\,\hat{c}_{-\vec{k},s} \\ \hat{P}\,\hat{d}_{\vec{k},s}\,\hat{P}^{-1} &= -\eta^P_{\psi}\,\hat{d}_{-\vec{k},s} \end{aligned}} \tag{316}$$

fermionic particles and the associated anti-particles have opposite parities

Remark: If Parity is a symmetry of a QFT, i.e. if

$$\hat{P}\mathcal{L}(\vec{r})\hat{P}^{-1} = \mathcal{L}(-\vec{r}) \quad (*) \tag{317}$$

then $\mathcal{L}(\vec{r})$ must be a scalar, which excludes certain types of interactions.

(*) this means the physics is the same in a spatially inverted system with $\vec{r} \to -\vec{r}$

(B) Charge conjugation

Transforms negative energy solutions to positive energy solutions under change of sign of charge.

It can also be applied to neutral fields and should therefore better be called "field conjugation"

We define charge conjugation by a unitary

(318) $$\hat{C}^{+} = \hat{C}^{-1}$$

action on a scalar field $\hat{\phi}$

(319) $$\boxed{\hat{C}\,\hat{\phi}(x)\,\hat{C}^{-1} = \eta^{C}_{\phi}\,\hat{\phi}^{+}(x)}$$

since $\hat{C}^{2} = 1 \quad \Rightarrow \quad \boxed{\eta^{C}_{\phi} = \pm 1}$

action on a vector field $\hat{\vec{A}}$

(320) $$\boxed{\hat{C}\,\hat{\vec{A}}(x)\,\hat{C}^{-1} = -\hat{\vec{A}}(x)}$$

since we want $\quad \vec{p} - e\vec{A} \xrightarrow{C} \vec{p} - e\vec{A}$

and $\quad C e C^{-1} \longrightarrow -e$

action on a bi-spinor field ψ

How to define the action on a bi-spinor? Consider (non-quantized) Dirac field in an external elm. field

$$(321)\quad i\partial_t \psi = \left(\alpha_j(-i\partial_j - eA^j) + eA^0 + \beta m\right)\psi$$

This equation has a symmetry

$$(322)\quad \psi \longrightarrow \psi^C = \underline{\underline{C}}\beta\psi^* \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

where

$$(323)\quad \boxed{\underline{\underline{C}} = \begin{pmatrix} 0 & i\sigma_y \\ -i\sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}$$

$$(324)\quad \boxed{\begin{aligned} &\underline{\underline{C}} = -\underline{\underline{C}}^T = -\underline{\underline{C}}^{-1} \qquad \underline{\underline{C}}^2 = -\mathbb{1} \\ &\underline{\underline{C}}\,\alpha_j\,\underline{\underline{C}}^{-1} = -\alpha_j^* \qquad \underline{\underline{C}}\beta\underline{\underline{C}}^{-1} = -\beta^* \end{aligned}}$$

<u>proof:</u> multiply complex conjugate of eq. (321) with $\underline{\underline{C}}\beta$:

$$-i\partial_t \psi^C = \underline{\underline{C}}\underbrace{\beta\left[\alpha_j^*\right.}_{-\alpha_j^*\beta}\left.(i\partial_j - eA^j) + eA^0 + \beta m\right]\psi^*$$

$$i\partial_t \psi^C = \left(\alpha_j(-i\partial_j + eA^j) - eA^0 + \beta m\right)\psi^C$$

which is identical to (327) with $e \to -e$

Thus we define

(325) $$\boxed{\hat{C}\hat{\psi}\hat{C}^{-1} = \eta^C_\psi \underline{\underline{C}}\beta\hat{\psi}^* = \eta^C_\psi \underline{\underline{C}}\,\bar{\hat{\psi}}^T}$$

note that $\hat{\psi}^* = (\hat{\psi}^+)^T \quad \beta\hat{\psi}^+ = \bar{\hat{\psi}}$

$\Rightarrow$ The free Dirac Lagrangian is invariant under charge conjugation

(326) $$\hat{C}\mathcal{L}(x)\hat{C}^+ = \mathcal{L}(x)$$

What is the action of $\hat{C}$ on annihilation and creation operators?

$\boxed{\hat{\phi}}$ Klein Gordon field (scalar)

remember: $$\hat{C}\hat{\phi}(x)\hat{C}^+ = \eta^C_\phi \hat{\phi}^+(x)$$

$$\Rightarrow \quad \eta^C_\phi \sum_{k_\mu} \frac{1}{\sqrt{2E_k L^3}}\left(\hat{a}^+_{\vec{k}} e^{-ik_\mu x^\mu} + \hat{c}_{\vec{k}} e^{+ik_\mu x^\mu}\right)$$

$$= \hat{C}\sum_{k_\mu} \frac{1}{\sqrt{2E_k L^3}}\left(\hat{a}_{\vec{k}} e^{-ik_\mu x^\mu} + \hat{c}_{\vec{k}} e^{ik_\mu x^\mu}\right)\hat{C}^+$$

i.e.

(327) $$\boxed{\begin{aligned} \hat{C}\,\hat{a}_{\vec{k}}\,\hat{C}^{+} &= \eta^{c}_{\phi}\,\hat{c}_{\vec{k}} \\ \hat{C}\,\hat{c}_{\vec{k}}\,\hat{C}^{+} &= \eta^{c}_{\phi}\,\hat{a}_{\vec{k}} \end{aligned}}$$

$\hat{C}$ interchanges positive and negative energy solutions without changing the momentum.

$\boxed{\hat{\vec{A}}}$ elm. field (vector)

$$\hat{C}\,\hat{\vec{A}}(x)\,\hat{C}^{+} = -\hat{\vec{A}}(x)$$

i.e.

$$\hat{C}\sum_{\vec{k},\lambda}\frac{1}{\sqrt{2\omega L^3}}\left(\hat{a}_{\vec{k},\lambda}\,\vec{e}^{\,\lambda}_{\vec{k}}\,e^{-ik_\mu x^\mu}+\hat{a}^{+}_{\vec{k},\lambda}\,\vec{e}^{\,\lambda *}_{\vec{k}}\,e^{ik_\mu x^\mu}\right)\hat{C}^{+}$$

$$= -\hat{\vec{A}}(x)$$

$\Rightarrow$ (328) $$\boxed{\hat{C}\,\hat{a}_{\vec{k},\lambda}\,\hat{C}^{+} = -\hat{a}_{\vec{k},\lambda}}$$

$\boxed{\hat{\psi}}$ Dirac field (bi-spinor)

positive energy solutions $$u(\vec{k},s) = \sqrt{E+m}\begin{pmatrix}1\\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\end{pmatrix}\chi^{(s)}$$

negative energy solutions

$$v(\vec{k},s) = \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m} \\ 1 \end{pmatrix} \left(-i\sigma_y \chi^{(s)}\right)$$

$$-i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (-i\sigma_y)^*$$

thus we find

$$\underline{\underline{C}}\,\beta\, v^* = \sqrt{E+m} \begin{pmatrix} 0 & -i\sigma_y \\ -i\sigma_y & 0 \end{pmatrix} \begin{pmatrix} \frac{\vec{\sigma}^*\cdot\vec{k}}{E+m} \\ 1 \end{pmatrix} \left(-i\sigma_y \chi^{(s)}\right)$$

$$= \sqrt{E+m} \begin{pmatrix} 1 \\ -\sigma_y \frac{\vec{\sigma}^*\cdot\vec{k}}{E+m} \sigma_y \end{pmatrix} \chi^{(s)} = u$$

note $\sigma_y \vec{\sigma}^* \sigma_y = -\vec{\sigma}$

i.e.

(329) $$\boxed{\begin{aligned} \underline{\underline{C}}\,\beta\, v^*(\vec{k},s) &= u(\vec{k},s) \\ \underline{\underline{C}}\,\beta\, u^*(\vec{k},s) &= v(\vec{k},s) \end{aligned}}$$

exchange of positive and negative energy solutions as in Klein Gordon field

so:

$$\hat{C}\hat{\psi}\hat{C}^{+} = \hat{C}\sum_{\vec{k},s} \frac{1}{\sqrt{2EL^3}} \Big(u(\vec{k},s) e^{-ik_\mu x^\mu} \hat{c}_{\vec{k},s} + v(\vec{k},s) e^{ik_\mu x^\mu} \hat{d}^{+}_{\vec{k},s} \Big) \hat{C}^{+}$$

$$= \eta^{C}_{\psi}\, C\beta\, \hat{\psi}^*$$

$$= \eta^C_{2\mu} C\beta \sum_{\bar{k},s} \frac{1}{\sqrt{2EL^3}} \Big(u^*(\bar{k},s)\, e^{ik_\mu x^\mu}\, \hat{c}^+_{\bar{k},s} + v^*(\bar{k},s)\, e^{-ik_\mu x^\mu}\, \hat{d}_{\bar{k},s} \Big)$$

$$= \eta^C_{2\mu} \sum_{\bar{k},s} \frac{1}{\sqrt{2EL^3}} \Big(v(\bar{k},s)\, e^{ik_\mu x^\mu}\, \hat{c}^+_{\bar{k},s} + u(\bar{k},s)\, e^{-ik_\mu x^\mu}\, \hat{d}_{\bar{k},s} \Big)$$

$\Rightarrow$

(330)
$$\boxed{\begin{aligned} \hat{C}\, \hat{c}_{\bar{k},s}\, \hat{C}^+ &= \eta^C_{2\mu}\, \hat{d}_{\bar{k},s} \\ \hat{C}\, \hat{d}_{\bar{k},s}\, \hat{C}^+ &= \eta^C_{2\mu}\, \hat{c}_{\bar{k},s} \end{aligned}}$$

- QED is invariant under charge conjugation

- application example

positronium decay

we will see that QED allows for the annihilation of an electron with its anti-particle the positron

$$e^+ + e^- \longrightarrow 2, 3, 4, \ldots \ \gamma \text{ quanta}$$

positronium

$e^+ + e^-$ → $^3S_1$ $\quad f^{(+)}(s,s') = f^{(+)}(s',s)$

symmetric

positronium → $^1S_0$ $\quad f^{(-)}(s,s') = -f^{(-)}(s',s)$

anti-symmetric

quantum state of positronium

$$|\Psi_0\rangle_{^3S_1} = \sum_{s,s'} \int d^3p \, f^{(+)}(p,s,s') \, \hat{c}^+_{p,s} \, \hat{d}^+_{-p,s'} |0\rangle$$

$$|\Psi_0\rangle_{^1S_0} = \sum_{s,s'} \int d^3p \, f^{(-)}(p,s,s') \, \hat{c}^+_{p,s} \, \hat{d}^+_{-p,s'} |0\rangle$$

and

$$\hat{C} |\Psi_0\rangle = \pm |\Psi_0\rangle \qquad \text{before decay}$$

since after decay we only have photons and

$$\hat{C} \hat{a}_{\vec{k},\lambda} \hat{C}^+ = -\hat{a}_{\vec{k},\lambda} \qquad n: \# \text{ of } \gamma \text{ quanta}$$

$$\Rightarrow \quad \hat{C} |\Psi_1\rangle = (-1)^n |\Psi_1\rangle \qquad \text{after decay}$$

triplet positronium $^3S_1 \longrightarrow 3\,(5,7,9\ldots)$ γ quanta

singlet positronium $^1S_0 \longrightarrow 2\,(4,6,8\ldots)$ γ quanta

(c) Time reversal

time evolution operator $\hat{U} = e^{-i\hat{H}t}$

(331) $$\hat{T}\hat{U}\hat{T}^{+} \stackrel{!}{=} \hat{U}^{-1}$$

Thus $\hat{T}$ cannot be unitary!

$\Rightarrow$ anti-unitary operator

unitary $|\psi\rangle \to \hat{U}|\psi\rangle$

$$\langle\phi|\psi\rangle \longrightarrow \langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle$$

anti unitary $|\psi\rangle \longrightarrow \hat{T}|\psi\rangle$

(332) $$\langle\phi|\psi\rangle \longrightarrow \langle\hat{T}\phi|\hat{T}\psi\rangle = \langle\phi|\psi\rangle^{*}$$

$\hat{T}$ acts as a unitary plus complex conjugation

(333) $$\boxed{\hat{T} = \hat{U}K}$$ $Kf = f^{*}$ complex conjugation

action of $\hat{T}$ on a scalar field $\hat{\phi}$

(334) $$\boxed{\hat{T}\hat{\phi}(\vec{r},t)\hat{T}^{-1} = \eta_{\phi}^{T}\hat{\phi}(\vec{r},-t)}$$

action of $\hat{T}$ on a vector field $\hat{A}^\mu(\vec{r},t)$

time reversal reverses current $\vec{j} \xrightarrow{T} -\vec{j}$

to make QED invariant under time reversal we should have

(335) $$\boxed{\hat{T}\hat{A}^\mu(\vec{r},t)\hat{T}^{-1} = \Lambda^\mu{}_\nu \hat{A}^\mu(\vec{r},-t)}$$

with $\Lambda^\mu{}_\nu$ from eq. (306) i.e.

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

action of $\hat{T}$ on bi-spinor field $\vec{\psi}(\vec{r},t)$

to have Dirac Lagrangian invariant under time reversal (and $t \to -t$) we choose

(336) $$\boxed{\hat{T}\vec{\psi}(\vec{r},t)\hat{T}^{-1} = \eta_\psi^T \underline{\underline{T}}\vec{\psi}(\vec{r},-t)}$$

where

(337) $$\boxed{\underline{\underline{T}} = \underline{\underline{C}}\gamma^5}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

then

(338) $$\hat{T}\mathcal{L}_{Dirac}(\vec{r},t)\hat{T}^{-1} = \mathcal{L}_{Dirac}(\vec{r},-t)$$

action of $\hat{T}$ on annihilation and creation operators

$\boxed{\hat{\phi}}$ **Klein-Gordon field (scalar field)**

$$\hat{T}\,\hat{\phi}(\vec{r},t)\,\hat{T}^{-1} = \eta_\phi^T\,\hat{\phi}(\vec{r},-t)$$

$$\overset{''}{} \sum_k \frac{1}{\sqrt{2EL^3}}\left(e^{ik_\mu x_\mu}\,\hat{T}\,\hat{a}_{\vec{k}}\,\hat{T}^{-1} + e^{-ik_\mu x_\mu}\,\hat{T}\,\hat{c}^{+}_{\vec{k}}\,\hat{T}^{-1}\right)$$

(remember: $\hat{T}\,i\,\hat{T}^{-1} = -i$)

$$= \eta_\phi^T \sum_k \frac{1}{\sqrt{2EL^3}}\left(e^{ik_\mu x_\mu}\,\hat{a}_{-\vec{k}} + e^{-ik_\mu x_\mu}\,\hat{c}^{+}_{-\vec{k}}\right)$$

where we have used $\vec{k} \to -\vec{k}$ to compensate "$-t$"

so: $k_\mu x^\mu \longrightarrow -k_\mu x^\mu$

$\Rightarrow$

(339)
$$\boxed{\begin{aligned} \hat{T}\,\hat{a}_{\vec{k}}\,\hat{T}^{-1} &= \eta_\phi^T\,\hat{a}_{-\vec{k}} \\ \hat{T}\,\hat{c}_{\vec{k}}\,\hat{T}^{-1} &= \eta_\phi^T\,\hat{c}_{-\vec{k}} \end{aligned}}$$

time reversal reverses direction of momentum as for parity

$\boxed{\hat{\vec{A}}(x)}$ elm. field

$$\hat{T}\hat{\vec{A}}(\vec{r},t)\hat{T}^{-1} = -\hat{\vec{A}}(\vec{r},-t)$$

$$= -\sum_{\vec{k},\lambda} \frac{1}{\sqrt{2\omega L^3}} \left(\vec{e}^{\,\lambda}_{-\vec{k}}\, \hat{a}_{-k,\lambda} e^{ik_\mu x^\mu} + \vec{e}^{\,\lambda *}_{-\vec{k}}\, \hat{a}^+_{-k,\lambda} e^{-ik_\mu x^\mu} \right)$$

where we have made $\vec{k} \to -\vec{k}$ to compensate $t \to -t$ in $k_\mu x^\mu$

$$\overset{!}{=} \sum_{\vec{k},\lambda} \frac{1}{\sqrt{2\omega L^3}} \Big(\vec{e}^{\,\lambda *}_{\vec{k}}\, \hat{T}\hat{a}_{\vec{k},\lambda}\hat{T}^{-1} e^{ik_\mu x^\mu} + \vec{e}^{\,\lambda}_{\vec{k}}\, \hat{T}\hat{a}^+_{\vec{k},\lambda}\hat{T}^{-1} e^{-ik_\mu x^\mu} \Big)$$

where we used $\hat{T}\vec{e}^{\,\lambda}_{\vec{k}}\hat{T}^{-1} = \vec{e}^{\,\lambda *}_{\vec{k}}$ and $\hat{T}\, i\, \hat{T}^{-1} = -i$

using circular polarization basis

$$\left(\vec{e}^{\,\pm}_{-\vec{k}}\right)^* = -\vec{e}^{\,\pm}_{\vec{k}}$$

$$(340) \quad \boxed{\hat{T}\,\hat{a}_{\vec{k},\pm}\,\hat{T}^{-1} = \hat{a}_{-\vec{k},\pm}}$$

time reversal of circular polarized photons reverses momentum

$\boxed{\hat{\psi}(x)}$ Dirac field

Here we have to first find the action of the $\underline{\underline{I}}$ matrix, eq. (337) on the bi-spinors.
After some algebra one finds

(347)
$$\boxed{\begin{aligned} \underline{\underline{I}}\, u(\vec{k},s) &= (-1)^{\frac{1}{2}-s}\, u^*(-\vec{k},-s) \\ \underline{\underline{I}}\, v(\vec{k},s) &= (-1)^{\frac{1}{2}-s}\, v^*(-\vec{k},-s) \end{aligned}}$$

Thus from $\hat{T}\hat{\psi}(\vec{r},t)\hat{T}^{-1} = \eta^T_\psi \underline{\underline{I}}\,\hat{\psi}(\vec{r},-t)$

$$\hat{T}\hat{\psi}\hat{T}^{-1} = \hat{T}\sum_{\vec{k},s}\frac{1}{\sqrt{2EL^3}}\Big(u(\vec{k},s)e^{-ik_\mu x^\mu}\hat{c}_{\vec{k},s} + v(\vec{k},s)e^{ik_\mu x^\mu}\hat{d}^+_{\vec{k},s}\Big)\hat{T}^{-1}$$

$$= \sum_{\vec{k},s}\frac{1}{\sqrt{2EL^3}}\Big(u^*(\vec{k},s)e^{ik_\mu x^\mu}\hat{T}\hat{c}_{\vec{k},s}\hat{T}^{-1} + v^*(\vec{k},s)e^{-ik_\mu x^\mu}\hat{T}\hat{d}^+_{\vec{k},s}\hat{T}^{-1}\Big)$$

$$\overset{!}{=} \eta^T_\psi\sum_{\vec{k},s}\frac{(-1)^{\frac{1}{2}+s}}{\sqrt{2EL^3}}\Big(u^*(\vec{k},s)e^{ik_\mu x^\mu}\hat{c}_{-\vec{k},-s} + v^*(\vec{k},s)e^{-ik_\mu x^\mu}\hat{d}^+_{-\vec{k},-s}\Big)$$

where we have used $\vec{k}\to-\vec{k}$ to compensate $-t$ and $s\to -s$ in the last line so

$$\boxed{\begin{aligned} \hat{T}\,\hat{c}_{\vec{k},s}\,\hat{T}^{-1} &= (-1)^{\frac{1}{2}+s}\,\hat{c}_{-\vec{k},-s} \\ \hat{T}\,\hat{d}_{\vec{k},s}\,\hat{T}^{-1} &= (-1)^{\frac{1}{2}+s}\,\hat{d}_{-\vec{k},-s} \end{aligned}} \tag{342}$$

or specifically

$$\boxed{\begin{aligned} \hat{T}\,\hat{c}_{\vec{k},\uparrow}\,\hat{T}^{-1} &= \hat{c}_{-\vec{k},\downarrow} \\ \hat{T}\,\hat{c}_{\vec{k},\downarrow}\,\hat{T}^{-1} &= -\hat{c}_{-\vec{k},\uparrow} \end{aligned}} \tag{343}$$

Time reversal reverses momentum and spin and gives one minus sign

(D) The PCT theorem

If the Lagrange density of a relativistic quantum field theory (Lorentz invariance) is hermitian, normal ordered and contains only fields quantized according the spin-statistic theorem, then

$$\hat{\Theta} = \hat{P}\hat{C}\hat{T}$$

is a symmetry of the theory

$$\boxed{\begin{aligned} \hat{O}\,\hat{\phi}(x)\,\hat{O}^{-1} &= \eta_\phi\, \hat{\phi}^{+}(-x) \\ \hat{O}\,\hat{\bar{\psi}}(x)\,\hat{O}^{-1} &= \eta_\psi\, \gamma^5 \gamma^0\, \bar{\psi}^T(-x) \\ \hat{O}\,\hat{A}^\mu(x)\,\hat{O}^{-1} &= -\hat{A}^\mu(-x) \end{aligned}} \qquad (344)$$

consequences:

> masses of stable particles and anti-particles are the same
>
> the same holds for unstable particles that do not decay into single-particle states

proof for stable particles

particle $\quad H|k\rangle = E_k |k\rangle \qquad E_k = \sqrt{m^2+k^2}$

anti-particle $\quad H|\bar{k}\rangle = \bar{E}_k |\bar{k}\rangle \qquad \bar{E}_k = \sqrt{\bar{m}^2+k^2}$

PCT : $\quad [\hat{O}, \hat{H}] = 0$

$$\hat{O}\hat{H}|k\rangle = E_k \hat{O}|k\rangle = E_k |\bar{k}\rangle$$

$$\overset{\shortparallel}{\hat{H}\hat{O}|k\rangle} = \hat{H}|\bar{k}\rangle = \bar{E}_k |\bar{k}\rangle \qquad \underline{m = \bar{m}} \quad \square$$